2:10 000, twice that in the first generation; and this proportion will afterwards have no tendency whatever to increase. If, on the other hand, brachydactyly were recessive, the proportion in the second generation would be 1:100020 001, or practically 1:100 000 000, and this proportion would afterwards have no tendency to decrease.

In a word, there is not the slightest foundation for the idea that a dominant character should show tendency to spread over a whole population, or that a recessive should tend to die out.

I ought perhaps to add a few words on the effect of the small deviations from the theoretical proportions which will, of course, occur in every generation. Such a distribution as $p_1:2q_1:r_1$, which satisfies the condition $q_1^2=p_1r_1$, we may call a *stable* distribution. In actual fact we shall obtain in the second generation not $p_1:2q_1:r_1$ but a slightly different distribution $p_1':2q_1':r_1'$, which is not 'stable'. This should, according to theory, give us in the third generation a 'stable' distribution $p_2:2q_2:r_2$, also differing slightly from $p_1:2q_1:r_1$, and so on. The sense in which the distribution $p_1:2q_1:r_1$ is 'stable' is this, that if we allow for the effect of casual deviations in any subsequent generation, we should, according to theory, obtain at the next generation a new 'stable' distribution differing but slightly from the original distribution.

I have, of course, considered only the very simplest hypotheses possible. Hypotheses other than that of purely random mating will give different results, and, of course, if, as appears to be the case sometimes, the character is not independent of that of sex, or has an influence on fertility, the whole question may be greatly complicated. But such complications seem to be irrelevant to the simple issue raised by Mr Yule's remarks.

P.S. I understand from Mr Punnett that he has submitted the substance of what I have said above to Mr Yule, and that the latter would accept it as a satisfactory answer to the difficulty that he raised. The 'stability' of the particular ratio 1:2:1 is recognised by Professor Karl Pearson (*Phil. Trans. Royal Society* (A) **203**, 60).

Chapter 7

Study of the diffusion equation with growth of the quantity of matter and its application to a biological problem

(*Étude de l'équation de la diffusion avec croissance de la quantité de la matière et son application à un problème biologique*)

by **A. KOLMOGOROV**, **I. PETROVSKY** and **N. PISKOUNOV**

From the original in French we give a translation of the first part of a remarkable paper by Russian mathematicians of whom Kolmogorov is perhaps the best known for the sheer versatility and penetration of his mathematical talent and who, like Feller, it would be unthinkable to omit from a list of outstanding contributors to Applicable Mathematics in general, and to probability theory in particular. The paper itself is well known in bio-mathematical circles and frequently cited in recent work concerned with genetic propagation and with the spatial dispersion of disease. The kind of solution examined is of interest also in neurophysiology in connection with the Hodgkin–Huxley equations and in certain models for traffic flow. Yet the paper does not seem easy to obtain.

The particular interest of the fragment here reprinted is its development of a special kind of solution of the the diffusion equation of stationary wave type, which could only arise when a forcing term is present. A similar study, slightly less general, and with a possibly more pedestrian analysis, was published at the same time by Fisher.† The connection with epidemics arises from assumptions concerning the spatial spread mechanism and the approximate reduction of the model to diffusion equation type in order to force a solution.

The work appeared originally in the first volume of an apparently now defunct foreign language edition of the Bulletin of the State University of Moscow in 1937, whose purpose was to diffuse to a wider international audience selections from work in the mathematical sciences pursued in the URSS.

†Fisher, R. A. The Wave of Advance of Advantageous Genes, *Ann. Eug.* London, 7 (1937) 355–369.

We have confined our detailed presentation to the first part of the Kolmogorov, Petrovsky and Piskounov paper for the possibly pusillanimous reason that the remainder is almost entirely existence proofs. However, a translation is included of the sequence of theorem statements without the proofs which, although of interest, could perhaps be shortened and, anyway, are of secondary relevance to the theme of this book. The translation is for reasons of clarity not entirely literal, and we remark that indeed the original suffers defects of repetitiveness and obscurity which do not, however, diminish its overall merit.

STUDY OF THE DIFFUSION EQUATION WITH GROWTH OF THE QUANTITY OF MATTER AND ITS APPLICATION TO A BIOLOGICAL PROBLEM

A. Kolmogorov, I. Petrovsky and N. Piskounov†

7.1 We start with the diffusion equation, considered for increased simplicity in two dimensions:

$$\frac{\partial v}{\partial t} = k\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right], \quad k > 0 . \tag{7.1}$$

x and y are the coordinates of a point in the plane, t is time, v is the density of matter at the point (x, y) at the instant t. We now suppose that, in addition to diffusion, growth of the quantity of matter takes place with a speed at a given place and time that depends on the density there. Then we have

$$\frac{\partial v}{\partial t} = k\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] + F(v) .\text{‡} \tag{7.2}$$

Of course we are interested only in values of $F(v)$ for which $v \geqslant 0$. We suppose in what follows that $F(v)$ is continuous and differentiable as often as necessary with respect to v, and that in addition it satisfies the conditions:

$$F(0) = F(1) = 0 ; \tag{7.3}$$

$$F(v) > 0 , \quad 0 < v < 1 ; \tag{7.4}$$

$$F'(0) = \alpha > 0 ; \quad F'(v) < \alpha , \quad (0 < v \leqslant 1) . \tag{7.5}$$

Thus we are assuming that when v is extremely small the speed $F(v)$ of growth of v is proportional to v with constant of proportionality α, and that moreover

† *Translators' Note:* This translation is of pp. 1–11 and part of p. 12. Translations are included of the statements of the seventeen theorems whose proof occupies pages 12–25. The original paper contains 25 pages in all.

‡ *Translators' Note:* The authors curiously enough do not say so here specifically, but $F(v)$ is, of course, the speed of growth of the matter, expressed as a time increase in the density v which is itself a function of t as well as of the space coordinates.

a state of 'saturation' arises as v approaches unity, at which time the growth of v ceases. Thus we shall consider only those solutions of (7.2) which satisfy the condition

$$0 \leqslant v \leqslant 1 . \tag{7.6}$$

The arbitrary initial values of v at $t = 0$ which satisfy (7.6) define a unique solution of (7.2) for $t > 0$, which satisfies (7.6).†

Next we suppose that the density is independent of y. Thus (7.2) reduces to

$$\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2} + F(v) . \tag{7.7}$$

Suppose now that, at $t = 0$, the density v is zero for $x < a$, while for $a \leqslant b < x$ it has its maximum value $v = 1$.

It is clear that as time t increases, values of density near unity propagate from right to left, pushing the lower densities ahead of them towards the left. In the particular case $a = b$, Fig. 7.1 shows roughly what happens.

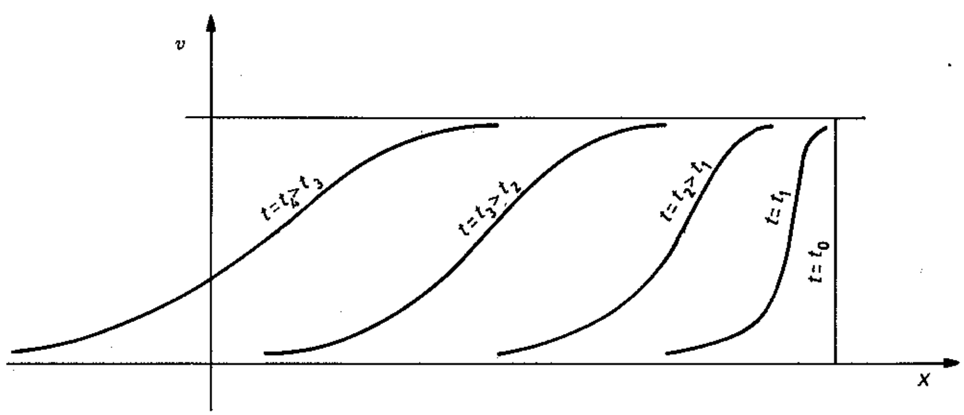

Fig. 7.1

The portion of the density versus space curve (v, x) corresponding to the main part of the decrease of the density from 1 to 0 is displaced with time from right to left. It is found that as $t \to \infty$ the curve tends to a limiting form. The problem is to determine this limiting form and the limiting velocity of its movement from right to left. It will be shown that the limiting velocity sought is given by

$$\lambda_0 = 2 \, [ka]^{\frac{1}{2}} , \tag{7.8}$$

†*Translators' Note:* This is proved in section 7.4 of the paper which is not included in its entirety.

and that the limiting form of the density curve is that solution of the equation

$$\lambda_0 \frac{dv}{dx} = k \frac{d^2 v}{dx^2} + F(v) , \tag{7.9}$$

which tends to zero as $x \to -\infty$, and to unity as $x \to +\infty$. Such a solution exists and is unchanged by a transformation $x' = x + c$. Note that (7.9) can be obtained as follows. We look for a solution of (7.7) which retains its shape as a function of x yet moves from right to left with speed λ. The form of such a solution will be

$$v(x, t) = v(x + \lambda t) . \tag{7.10}$$

Regarding now v as a function of the single variable $z = x + \lambda t$ we obtain

$$\lambda \frac{dv}{dz} = k \frac{d^2 v}{dz^2} + F(v) .$$

This equation, as we shall see, admits solutions for all $\lambda \geqslant \lambda_0$ which satisfy the conditions mentioned in connection with [solutions of] (7.9). But it is only for $\lambda = \lambda_0$ that we obtain the limiting form which concerns us of the density curve under the initial conditions indicated above. To understand the fact, which may at first sight appear strange, of the existence of solutions of (7.7) of the form (7.10) for $\lambda > \lambda_0$, solutions for which the movement of the region of high (near unity) densities occurs with a speed in excess of λ_0, let us examine the limiting case $k = 0$. Then there is no diffusion and (7.7) is easily integrated. Under our initial conditions, for values of $x < a$ where the density was initially zero it will remain zero for all $t > 0$. However, it is easy to calculate that solutions of (7.7) of the form (7.10), satisfying all the conditions specified above, can be found for all $\lambda > 0$. The apparent motion of matter from right to left is caused in reality [merely] by the increase in density at each point which occurs independently of what happens elsewhere.

The facts expounded in this introduction find in section 7.2 an application to some biological problems.† Proofs are given in sections 7.3 and 7.4.

7.2 We now consider a certain territory, or region, populated by some arbitrary species. First let us suppose that a dominant gene A is distributed over the territory, having constant concentration p $(0 \leqslant p \leqslant 1)$. Let us suppose next that individuals with characteristic A (that is, belonging to genotypes AA and Aa) have an advantage in their struggle for existence against individuals who do not possess characteristic A (belonging to genotype aa). More specifically, we shall assume that the ratio of the probability of survival of an individual with charac-

†*Translators' Note:* Actually one single biological problem.

teristic A to the probability of survival of an individual without it, is $1 + \alpha$, where α is a small positive number. Then up to terms of order α^2 we shall obtain for the increase in the concentration due to a single generation the expression†

$$\Delta p = \alpha p\,[1-p]^2\,. \tag{7.11}$$

Now let us assume that the concentration p varies across the territory occupied by the species under consideration, that is, that p depends on the x and y coordinates of the point in the plane. If then the individuals of our species remained immobile at fixed points, (7.11) would still hold. However, let us assume that in the interval between birth and reproduction each individual may move in an arbitrary direction (all directions having the same probability) through a distance having probability $f(r)\,\mathrm{d}r$ of a value between r and $r + \mathrm{d}r$ with

$$\rho = \left[\int_0^\infty r^2 f(r)\,\mathrm{d}r\right]^{\frac{1}{2}},$$

the mean square displacement. We then obtain instead of (7.11) the formula

$$\Delta p(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\xi,\eta)\frac{f(r)}{2\pi r}\,\mathrm{d}\xi\,\mathrm{d}\eta - p(x,y) + \alpha p(x,y)\,[1-p(x,y)]^2\,, \tag{7.12}$$

where $r = \{[x-\xi]^2 + [y-\eta]^2\}^{\frac{1}{2}}$.

We now suppose that p may be differentiated with respect to x, y and t (counting t by generations as the unit), that α and ρ are very small, and that the third moment

$$d^3 = \int_0^\infty r^3 f(r)\,\mathrm{d}r$$

is small by comparison with ρ^2. In this case we expand $p(\xi,\eta)$ in (7.12) as a Taylor series in $(\xi - x)$ and $(\eta - y)$, restricting ourselves to terms of the second order (terms of the first order are vanishingly small). This gives the following approximate differential equation for p:

$$\frac{\partial p}{\partial t} = \tfrac{1}{4}\rho^2\left[\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right] + \alpha p\,[1-p]^2\,{}^{\ddagger}. \tag{7.13}$$

†R. A. Fisher, *The Genetical Theory of Natural Selection*, Oxford Univ. Press (1930).
‡Concerning the passage from (7.12) to (7.13), see analogous consideration in A. Khintchine: *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*. Springer (1933).

All the considerations discussed in relation to the more general equation (7.2) apply to (7.13).

We now recall the hypotheses that were made. We supposed that the concentration p varies continuously as a function of space and time (and is differentiable with respect to x, y and t), and that the variations arise through [genetic] selection in the ratio $1 + \alpha:1$ to the advantage of the dominant criterion A, and also by the chance displacement of particular individuals in such a way that the mean square displacement between birth and reproduction of a single individual is ρ. Finally it was assumed that α and ρ are small (ρ in relation to distances over which significant changes in p may occur). Under these hypotheses, and measuring time by generations, we obtain equation (7.13).

Let us now examine the case of a vast region already occupied by the gene A with concentration p close to unity. Naturally a band of intermediate concentration will be found along the boundary of the region. We suppose that beyond this band p is close to zero. Since there is positive selection the region occupied by A will enlarge: in other words its boundary will move towards regions still unoccupied by A, yet all along the boundary of the region there will always exist a band of intermediate concentration. Our first problem is to determine the speed of propagation of the gene A, that is to say the speed of movement of the boundary. Formula (7.8) gives an immediate answer. Since $k = \frac{1}{4}\rho^2$ in this case the speed sought is

$$\lambda = \rho\,[\alpha]^{\frac{1}{2}}\,. \tag{7.14}$$

The second question naturally concerns the determination of the width of the intermediate band. By (7.9) the concentration p satisfies the equation

$$\lambda\frac{\mathrm{d}p}{\mathrm{d}n} = \tfrac{1}{4}\rho^2\frac{\mathrm{d}^2 p}{\mathrm{d}n^2} + \alpha p\,[1-p]^2$$

in the direction of the normal to the boundary. By using (7.14) this may be written as

$$\rho\alpha^{-\frac{1}{2}}\frac{\mathrm{d}p}{\mathrm{d}n} = \tfrac{1}{4}\frac{\rho^2}{\alpha}\frac{\mathrm{d}^2 p}{\mathrm{d}n^2} + p\,[1-p]^2\,.$$

Introducing the new variable

$$\nu = n\,[\alpha]^{\frac{1}{2}}/\rho \tag{7.15}$$

we obtain

$$\frac{\mathrm{d}p}{\mathrm{d}\nu} = \tfrac{1}{4}\frac{\mathrm{d}^2 p}{\mathrm{d}\nu^2} + p\,[1-p]^2\,, \tag{7.16}$$

which contains neither α nor ρ. The limiting conditions are the same as for (7.9), namely

$$p(+\infty) = 1, \qquad p(-\infty) = 0 .$$

From (7.15) we conclude that the width of the intermediate band is proportional to

$$L = \rho\,[\alpha]^{-\frac{1}{2}} . \tag{7.17}$$

Taking, for example, $\rho = 1, \alpha = 10^{-4}$ we get $\lambda = 0.01, L = 100$.

7.3 In this section we consider the equation

$$\lambda \frac{dv}{dx} = k \frac{d^2 v}{dx^2} + F(v) , \tag{7.18}$$

where λ and k are positive and $F(v)$ satisfies the conditions specified in the introduction. Our purpose is to find the relation between λ, k and $\alpha = F'(0)$ such that (7.18) has a solution satisfying

$$0 \leqslant v(x) \leqslant 1 ,$$

$$\lim_{x \to +\infty} v(x) = 1 \text{ and } \lim_{x \to -\infty} v(x) = 0 .$$

Let $p = dv/dx$. Then $d^2v/dx^2 = p\,dp/dv$ and on substituting into (7.18) we find that

$$\frac{dp}{dv} = \frac{\lambda p - F(v)}{kp} . \tag{7.19}$$

Our concern is with the integral curves of this equation which pass in the (p, v) plane between the straight lines $v = 0$ and $v = 1$. In general, among these integral curves there may exist members of the following types:

(i) Integral curves which approach neither $v = 0$ nor $v = 1$ within distance less than some $\epsilon > 0$;
(ii) Integral curves infinitely distant from the v-axis but which approach one of $v = 0$ or $v = 1$ asymptotically;
(iii) Integral curves which cut one of the lines $v = 0$, $v = 1$ at a finite distance from the v-axis;
(iv) Integral curves belonging to none of the preceding types and which pass through the points $v = 0, p = 0$, and $v = 1, p = 0$.

It is easy to see that no integral curve of type (i) can correspond to a solution of (7.18) of the type required, namely satisfying the stated conditions, since for such curves v can never take the values 0 and 1.

Integral curves of type (ii) do not in general exist since they would necessarily have to pass through points for which $|dp/dv|$ is very large. But on the type (ii) since $F(v)$ is bounded on (0,1) the ratio $[\lambda p - F(v)]/kp$ tends to λ/k for very large $|p|$.†

† *Translators' Note:* Implying, of course, a contradiction.

To integral curves of type (iii) correspond solutions of (7.18) which do not remain always between 0 and 1. The reason is as follows. Let us suppose for example that some curve of this type approaches the point $v = 1, p = p_1 \neq 0$. Near the line $v = 1$, $dp/dv \sim \lambda/k \neq 0$.

Consequently we can regard p as a function of v, $p = \phi(v)$ say. Since $\phi(1) = p_1 \neq 0$, over a small interval $1 - \epsilon < v < 1 + \epsilon$, $|\phi(v)|$ must be greater than a certain positive constant C. Let x_0 be the value of x for which $v = 1 - \epsilon$. Then integrating the equation $dv/dx = \phi(v)$ we get

$$\int_{x_0}^{x} dx = x - x_0 = \int_{1-\epsilon}^{v} \frac{dv}{\phi(v)} .$$

It follows that as v increases from $1 - \epsilon$ to $1 + \epsilon$ the absolute change in x does not exceed $2\,\epsilon/C$. Thus, as x increases from x_0 to $x_0 + 2\,\epsilon/C$, v necessarily passes through the value 1.

It remains to examine the integral curves of type (iv). Each of the points $v = 0, p = 0$ and $v = 1, p = 0$, is a singular point of the differential equation (7.19). An integral curve of type (iv) must approach each of these points without having already intersected the lines $v = 0$ and $v = 1$ for other values of p. Thus if such curves exist the characteristic equation for each point must have real roots. We express $F(v)$ in the form

$$F(v) = \alpha v + \phi_1(v) .$$

Then clearly $\phi_1(v) = o(v)$. Thus the characteristic equation at $v = 0, p = 0$ is of the form

$$\begin{vmatrix} \lambda - \rho & -\alpha \\ k & -\rho \end{vmatrix} = 0 ,$$

or

$$\rho^2 - \lambda\rho + \alpha k = 0 . \tag{7.20}$$

To ensure that the equation has real roots we must have

$$\lambda^2 \geqslant 4\alpha k^{\dagger} .$$

To obtain the characteristic equation at $v = 1, p = 0$ we make a change of variables, putting $v = 1 - u$. Then

$$\frac{dp}{du} = \frac{-\lambda p + \phi(u)}{kp} ,$$

where $\phi(u) = F(1 - u)$.

† *Translator's Note:* Original omits factor 4.

It is evident that $F'(1) \leqslant 0$† and hence $\phi'(0) = -F'(1) = A \geqslant 0$. Consequently

$$\phi(u) = Au + o(u) ,$$

and the characteristic equation at $v = 1, p = 0$ has the form

$$\begin{vmatrix} -\lambda - \rho & A \\ k & -\rho \end{vmatrix} = 0$$

or

$$\rho^2 + \lambda\rho - Ak = 0 . \tag{7.21}$$

To have real roots it is necessary that $\lambda^2 \geqslant -4Ak$. Since $\alpha > 0$, (7.20) has real roots of the same sign. Hence (0, 0) is a node. All curves close enough to this point pass through it. When $A > 0$, (7.21) has roots of opposite sign. In this case only two integral curves pass through the point $v = 1, p = 0$ in exactly determined directions, that is to say, directions given by

$$m_1 u + n_1 p = 0 , \quad m_2 u + n_2 p = 0 . \tag{7.22}$$

It is known‡ that the coefficients m_1, n_1, m_2, n_2 are determined by the equations

$$km_1 - \rho_1 n_1 = 0 , \quad km_2 - \rho_2 n_2 = 0 , \tag{7.23}$$

ρ_1 and ρ_2 being roots of characteristic equation (7.21). Since the roots have different signs the slopes of the lines (7.22) have different signs too*. This is why in each of the angles formed by the intersection of the lines $v = 1, p = 0$, there passes only one integral curve of (7.19), traversing the point $v = 1, p = 0$. Fig. 7.2 shows schematically the arrangement of these curves

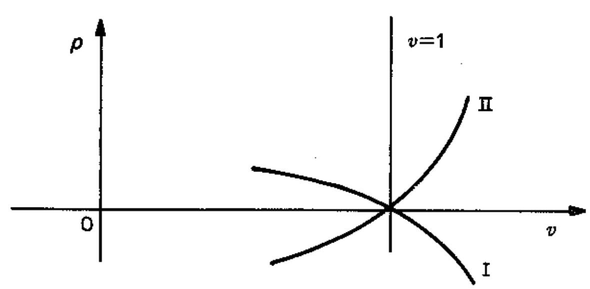

Fig. 7.2

† *Translators' Note:* From the original conditions that F has to satisfy.
‡ See, for example, Bendixon, *Acta Matematica* 1, 24 (1900).
* It can be shown moreover that the two slopes of the tangents to the integral curves of (7.19) at the origin are positive.

Curve II cuts the p-axis below the origin. For (7.19) shows that $dp/dv > 0$ in that part of the plane between $v = 0$ and $v = 1$, and below the v-axis. Thus curve II must be excluded from consideration. It remains to investigate curve I†.

We intend to show that curves I intersect the p-axis at the origin. Let us show that these curves cannot cut the p-axis below the origin. For this purpose we consider the isoclines of (7.19). The equation of the family has the form

$$\frac{\lambda p - F(v)}{kp} = C , \tag{7.24}$$

C being the value of dp/dv at (v, p). Thus

$$p = \frac{F(v)}{\lambda - Ck} . \tag{7.24'}$$

Relation (7.24) represents a family of curves passing through (0, 0) and (1, 0). The family is shown schematically in Fig. 7.3.

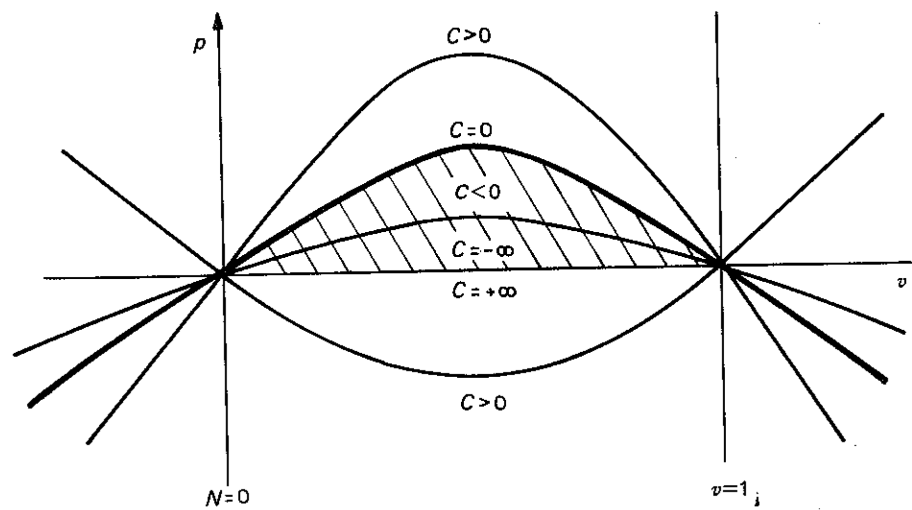

Fig. 7.3

Next to each curve is shown the value of C to which it corresponds. Th heavy curve corresponds to $C = 0$. The higher the maxima of the curves th higher the value of C, tending to the value λ/k which corresponds to bot $v = 0$ and $v = 1$. In the region enclosed between the $C = 0$ curve and the v-axi

†If $A = 0$ one can only assert that there exists at least one integral curve of type I whic tends to (1, 0) with negative slope. (See I. Petrovsky, Über das Verhalten der Integra kurven eines Systems gewöhnlicher Differentialgleichungen. *Récuil Mathématique,* 4 107–156 (1934)).

$C < 0$, and C is moreover very large in absolute value in the neighbourhood of the v-axis. Below the v-axis $C > 0$ and as the minima of the curves decrease so C decreases from $+\infty$ to λ/k.

Now it can easily be seen that the integral curve of type I (see Fig. 7.2) cannot intersect the p-axis below the origin. For if this were the case curve I would have to intersect the v-axis. But since dp/dv is $-\infty$ immediately above this axis and $+\infty$ immediately below, the convexity of I at the point of intersection with the v-axis [would be] oriented towards the line $v = 1$. Now, for the curve to reach the point $(1, 0)$ it would be necessary for dp/dv to be [positive] infinite above the v-axis, which is impossible. Hence integral curves I cannot intersect $v = 0$ below the v-axis.†

Next we show that integral curve I cannot cut the p-axis above the origin. For this it suffices to prove the existence of a half-line through the origin, crossing the first quadrant, which is not intersected by any integral curve cutting the positive p-axis. We now pass to the proof of the existence of such a half-line.

From (7.24) we have

$$\left(\frac{d\bar{p}}{dv}\right)_{v=0} = \frac{\alpha}{\lambda - Ck}$$

where $d\bar{p}/dv$ is the derivative of $p = p(v)$ defined by (7.24').

Now let us determine C such that $(d\bar{p}/dv)_{v=0} = C$. We get

$$\frac{\alpha}{\lambda - Ck} = C ,$$

or

$$C^2 - C\lambda + \alpha = 0,$$

giving

$$C = \frac{\lambda \pm [\lambda^2 - 4\alpha k]^{\frac{1}{2}}}{2k} . \tag{7.25}$$

Since we have assumed that $\lambda^2 \geqslant 4\alpha k$ the two values of C given by (7.25) are real and positive. Let us now make a specific choice and, denoting it by C_0, draw a line

$$p = C_0 v . \tag{7.26}$$

It is easily seen that for all points of the sector between $v = 0$ and $v = 1$ that lie above, or on, (7.26) (excluding only the origin) we have $dp/dv > C_0$.‡

†*Translators' Note:* We have amended this passage slightly, since in the original, owing to uncorrected misprints no doubt, it does not appear to make complete sense.

‡ p is related to v by (7.19).

Hence, no integral curve passing through an arbitrary point of the p-axis above the origin will ever intersect that part of (7.26) located above the v-axis. Thus we have proved that each curve of type I (see Fig. 7.2) passes through the origin.

We now prove that there exists merely one integral curve of type I. (The only case where this is not necessary is when the slope A at $(1, 0)$ is zero).

It has been shown that every curve of type I passes through the origin. Moreover it follows from (7.19) that for fixed v, dp/dv increases as $p (> 0)$ increases. Thus two integral curves passing through the origin cannot pass through $(1, 0)$.

Finally we show that to curve I corresponds a solution of equation (7.18) satisfying the conditions stipulated at the outset. Note first that no perpendicular to the v-axis cuts integral curve I in more than one point for, if it did, dp/dv would have to be infinite above the v-axis. We take p a function of v, that is $p = \phi(v)$, along the curve. Recalling that curve I cuts the v-axis at $(1, 0)$ at a negative angle, and the origin at a positive angle we write for small values of v,

$$p = k_1 v + o(v) , \tag{7.27}$$

and for small values of $1 - v$,

$$p = k_2 [1 - v] + o(1 - v) , \tag{7.28}$$

where k_1 and k_2 are positive.

Since $p = dv/dx$ we have $dv/dx = \phi(v)$, or $dx = dv/\phi(v)$. Integrating we obtain

$$x - x_0 = \int_{v_0}^{v} \frac{dv}{\phi(v)} , \qquad 0 < v_0 < 1 .$$

By virtue of (7.27) and (7.28) it follows that as $v \to 0, x \to -\infty$, and as $v \to 1, x \to +\infty$, which was to be proved.

7.4 Instead of equation (7.7) of which we spoke in the introduction we shall consider in this section the equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = F(v) , \tag{7.29}$$

where $F(v)$ satisfies the following conditions:

$$F(0) = F(1) = 0 , \tag{7.30}$$

$$F(v) > 0 , \qquad 0 < v < 1 , \tag{7.31}$$

$$F'(0) = 1 ; \tag{7.32}$$

$$F'(v) < 1 \quad \text{for} \quad 0 < v < 1 \; . \tag{7.33}$$

$F'(v)$ is bounded and continuous on the interval (0, 1). Moreover it is supposed that $F(v)$ is differentiable several times. Note that the general form (7.7) can always be reduced to form (7.29) by the change of variables $x = [k/\alpha]^{\frac{1}{2}}\,\bar{x}$, $t = \bar{t}/\alpha$.

The main objective of this section is to prove that as $t \to +\infty$ the portion of the density curve $v(x, t)$ (as a function of x) corresponding to the main part of the reduction from 1 to 0 moves with time to the left with a speed that approaches 2 as a lower limit, and that the form of the density curve tends to the solution curve $u(x)$ of the equation

$$\frac{d^2u}{dx^2} - 2\frac{du}{dx} + F(u) = 0 \; , \tag{7.34}$$

which becomes zero for $x \to -\infty$ and 1 for $x \to +\infty$. The existence of such a solution was shown in section 7.2.

Before carrying out the proof of the fundamental theorems of this section we shall demonstrate the existence of a solution of the equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = F(x, t, v) \; ,$$

of which (7.29) is a special case, taking given values for $t = 0$, and we shall study its properties.

7.5 APPENDIX†

Theorem 1. In the equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = F(x, t, v) \; , \tag{7.35}$$

let the continuous, bounded function $F(x, t, v)$ satisfy the Lipschitz condition

$$|F(x_2, t, v_2) - F(x_1, t, v_1)| < k|x_2 - x_1| + k|v_2 - v_1| \; , \tag{7.36}$$

where k is a constant, independent of x, t and v. Let $f(x)$ be a bounded function, defined for all x. For increased simplicity it will be supposed that $f(x)$ possesses only a finite number of discontinuities. Then there exists a unique bounded function $v(x, t)$ for bounded t which, for $t > 0$, satisfies (7.35) and which for $t = 0$ is equal to $f(x)$ at all the points of continuity of that function.

† *Translators' Note:* The remainder of the paper contains 17 theorems and proofs. Here we give paraphrased statements of the theorems.

Theorem 2. Replacing $F(x, t, v)$ by another function $F_1(x, t, v)$ such that everywhere $F_1(x, t, v) \geqslant F(x, t, v)$ does not diminish $v(x, t)$ provided that the initial conditions do not change.

Remark. With the interpretation of (7.35) as a heat propagation equation, $F(x, t, v)$ represents the intensity of heat creation and the truth of Theorem 2 is evident physically.

Theorem 3. $v(x, t)$ cannot diminish if $f(x)$ increases.

Remark. A similar physical interpretation to that of Theorem 2 makes this theorem evident. In a bar $f(x)$ would be the initial temperature distribution along the bar.

Theorem 4. If $f(x) \geqslant 0$ everywhere and $F(x, t, 0) = 0$ then $v(x, t) \geqslant 0$.

Theorem 5. If in addition $f(x) > 0$ on a certain positive interval then for $t > 0$, $v(x, t) > 0$.

Theorem 6. If $F(x, t, 1) = 0$ and $f(x) \leqslant 1$, then $v(x, t) \leqslant 1$.

Theorem 7. If $v(x, t)$ is identical for $t = 0$ with the increasing differentiable function $f(x)$ and if for $t > 0$, it satisfies

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = F(t, v) \; , \tag{7.45}$$

then $v(x, t)$ is a non-decreasing function of x for $t > 0$.

Theorem 8. If $f^{(\epsilon)}(x) \to f(x)^{(0)}$ as $\epsilon \to 0$ in such a way that

$$\int_{-\infty}^{\infty} |f^{(\epsilon)}(x) - f^{(0)}(x)|\, dx \to 0 \; ,$$

then for each $t > 0$, the function $v^{(\epsilon)}(x, t)$ satisfying (7.35) for $t > 0$ and reducing to $f^{(\epsilon)}(x)$ for $t = 0$, tends to the function $v^{(0)}(x, t)$, also satisfying (7.35) and reducing to $f^{(0)}(x)$ when $t = 0$.

Theorem 9. The function $v(x, t)$ which satisfies (7.45) for $t > 0$, vanishes for $t = 0, x < 0$, and becomes unity for $t = 0, x > 0$, is a non-decreasing function of x, and $v(x, t) > 0$ for $t > 0$.

Theorem 10. For every fixed $x < 0$

$$\lim_{t \to \infty} v(x - 2t, t) = 0 \; .$$

Theorem 11. For t fixed let $v'_x\,(x, t)$ be regarded as a function of v (justified by Theorem 9). Write $v'_x\,(x, t) = \psi(v, t)$. Then for fixed v, ψ cannot be an increasing function of t.

Theorem 12. For arbitrary fixed t we have

$$v'_x(x, t) \geqslant u'(x)$$

if $v(x, t) = u(x)$. Here $u(x)$ is that solution of (7.34) to which reference was made at the beginning of section 7.3.

Theorem 13. Define $v^*(x, t)$ by

$$v^*(x, t) = v(x + \phi(t), t),$$

where $\phi(t)$ is so chosen that

$$v^*(0, t) = C = \text{constant}$$

everywhere. Then

$$\lim_{t\to\infty} v^*(x, t) = v^*(x) .$$

Theorem 14. As $t_0 \to +\infty$ the function

$$v_{t_0}(x, t) = v(x + \phi(t_0), t + t_0)$$

converges uniformly in $t \leqslant T =$ constant to a certain solution $v(x, t)$ of (7.29). $\phi(t_0)$ is defined to be such that $v_{t_0}(0, t_0) = C =$ constant for all t_0.

Theorem 15. As $t_0 \to +\infty$ the first-order partial derivatives of $v_{t_0}(x, t)$ tend to the corresponding partial derivatives of $\overline{v}(x, t)$ uniformly in every domain $\epsilon < t < T$, ϵ and T being arbitrary positive constants.

Theorem 16. Let $v_{t_0}(x, t)$ and $\overline{v}(x, t)$ have the constant value C along the curves $x = \phi_{t_0}(x)$ and $x = \phi(t)$ respectively. Then uniformly with respect to t for $\epsilon < t < T$

$$\lim_{t\to\infty} \phi'_{t_0}(t) = \phi'(t) .$$

Theorem 17. For arbitrary t we have

$$v(x, t) = u(x + 2)$$

and $d\phi/dv \to -2$ as $t \to \infty$.

PART II: MEDICAL APPLICATIONS

Chapter 8

Introduction to medical applications

The two chapters comprising Part II have a strong claim to be the most remarkable of the whole book. This can partly be ascribed to the fact that neither of the authors professed mathematics as a primary discipline, yet they both possessed powers of modelling, analysis, and deduction that many professionals would do well to emulate. This contrasts with Part 1 where the authors were exclusively professional mathematicians.

A. G. McKendrick (it is sometimes written (M'Kendrick), the author of Chapter 9 and co-author of Chapter 10, lived from 1876 to 1943. He began his career as an army doctor in the Indian Medical Service in about 1901. Details of his life and many-sided attributes can be found in an obituary by W. F. Harvey in the *Edinburgh Medical Journal*, **50**, 500-506, (1943). McKendrick appears to have been a self-taught mathematician. His lifelong interest in epidemiology may have been stimulated by service with Sir Ronald Ross in Sierra Leone in his early years, but it was undoubtedly the contact with disease in India that supplied the urge to expose and explain the mechanisms by which illnesses are propagated. The second part of his professional career began in 1920 when he was appointed Superintendent of the Laboratory of the Royal College of Physicians in Edinburgh, where he remained until retirement in 1941. From our point of view the most striking feature of this period is that a superintendent, encumbered by the administrative problems of a busy and expanding medical laboratory, could find the time to establish and lead a new research department in whose work he would himself participate, playing, indeed, a leading and distinguished role through his mathematical investigations. Both papers in this part belong to this period, though we shall refer below to an earlier key paper reporting investigations belonging to his army period.

W. O. Kermack (1898-1970), whose power as a mathematician is not entirely displayed by the second paper, where he is first author, is another remarkable figure. He was by profession a biochemist and lost his sight in a laboratory accident in 1925. An account of his life can be found in the Biographical Notices of Fellows of the Royal Society (Volume 17) to which he was