CHAPTER 11 - REMINDER SHEET

The subject of Chapter 11 are higher-order equations and applying first order concepts to it. A higher order differential equation is for example a second order equation of the form

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = F(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}).$$

If F does not explicitly contain y, then we can reduce the equation to one of first order by the substitution $v = \frac{\mathrm{d}y}{\mathrm{d}x}$. Indeed if v is of this form, then $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}v = \frac{\mathrm{d}v}{\mathrm{d}x}$, hence the above equation becomes

$$\frac{\mathrm{d}v}{\mathrm{d}x} = F(x, v).$$

If we obtain a solution $v(x)$ of this equation, then the solution y of the original equation is given as

$$y(x) = \int v(x)dx.$$

If instead x is not explicitly part of F, then again we can use the substitution $v = \frac{\mathrm{d}y}{\mathrm{d}x}$ to bring the original equation into the form of a first order differential equation, where y is now the independent variable. As y is a function of x, we need to apply the chain rule to get

$$\frac{\mathrm{d}v}{\mathrm{d}x} = v\frac{\mathrm{d}v}{\mathrm{d}y}.$$

Then we can solve the equation

$$v\frac{\mathrm{d}v}{\mathrm{d}y} = F(y, v)$$

for v and again integrate to get a solution of the original equation.

Example: Let

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (\frac{\mathrm{d}y}{\mathrm{d}x})^2.$$

Then by inspection we see that no y-term is involved. Thus the substitution $v = \frac{\mathrm{d}y}{\mathrm{d}x}$ gives the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}x} = v^2.$$

This is a separable first order linear differential equation with the general solution

$$v(x) = \frac{1}{C_1 - x}.$$

Hence the solution of the original equation is given by

$$y(x) = \int v(x)dx = -\ln(C_1 - x) + C_2.$$