CHAPTER 13 - REMINDER SHEET

The subject of Chapter 13 are solutions to homogeneous linear differential equations

$$a_0(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_1(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ldots + a_n(x)y = 0.$$

For simplicity say the differential equation is of order 2:

$$a_0(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_2(x)y = 0.$$

We assume that a_0, a_1 and a_2 are continuous, and that we solve the differential equation on an interval such that a_0 is never 0. The main theorem of the chapter then states, that any pair y_1 and y_2 of linearly independent particular solutions form a fundamental set of solutions, i.e. a general solution y is given as

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

for constants c_1 and c_2. These constants are uniquely determined by initial conditions $y(x_0) = A$ and $y'(x_0) = B$.

Example: Consider

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - y = 0.$$

Then we can solve this over the entire real line, as the coefficient of the second derivative never vanishes. One solution is clearly given as e^x and another solution is given as e^{-x}. These are linearly independent, hence a general solution of the differential equation is given as

$$y(x) = c_1 e^x + c_2 e^{-x}.$$

If we are given initial conditions $y(0) = 1$ and $y'(0) = 1$, then we get the equations

$$c_1 + c_2 = 1 \qquad c_1 - c_2 = 1,$$

which we solve for $c_1 = 1$ and $c_2 = 0$. Hence the unique solution to the initial value problem is given by e^x.

If we want to determine in general if a set of k functions $y_1(x), \ldots, y_k(x)$ is linearly independent, we can compute the so called Wronskian determinant, i.e the determinant of the $k \times k$ matrix, whose (i,j)th entry is given as the $i-th$derivative of the y_j. In our case the Wronskian determinant is given as

$$det\left(\begin{bmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{bmatrix}\right) = -1 - 1 = -2 \neq 0.$$

If the Wronskian is unequal to 0, then the functions are linearly independent.