CHAPTER 17 - REMINDER SHEET

The subject of Chapter 15 were solutions to second-order homogeneous linear differential equations with constant coefficients, i.e. differential equations of the form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0,$$

where $a, b, c \in \mathbb{R}$ are constants such that $a \neq 0$. We can generalize the characteristic polynomial approach to solving these differential equations easily to the case of nth order homogeneous linear equations with constant coefficients, i.e. the case of

$$a_n\frac{\mathrm{d}^ny}{\mathrm{d}x^n} + a_{n-1}\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} + \ldots + a_0y = 0.$$

Then the *characteristic polynomial* for this equation is given as

$$a_nr^n + a_{n-1}r^{n-1} + \ldots + a_0 = 0.$$

Like in the case of order two, we should factor the polynomial completely into linear factors. There are four cases to consider:

(1) The polynomial factors into a set of distinct real roots $\{r_1, \ldots, r_n\}$. Then

$$\{e^{r_1x}, \ldots, e^{r_nx}\}$$

is a fundamental set of solutions.

(2) Any real root r_i repeats with multiplicity k. Then we need to add the set of solutions

$$\{xe^{r_ix}, x^2e^{r_ix}, \ldots, x^{k-1}e^{r_ix}\}$$

to the fundamental set of solutions.

(3) There are distinct complex roots $r_1 = \lambda_1 + i\omega_1$ up to $r_n = \lambda_n + i\omega_n$. Then all their complex conjugates are also roots and fundamental solutions come in pairs

$$\{e^{(\lambda_k+i\omega_k)x}, e^{(\lambda_k-i\omega_k)x}\}.$$

(4) There is a repeated complex root $r_k = \lambda_k + i\omega_k$. Similarly to the case of repeated real roots we need to add the solutions

$$\{xe^{r_kx}, x^2e^{r_kx}, \ldots, x^{k-1}e^{r_kx}, xe^{\overline{r_k}x}, x^2e^{\overline{r_k}x}, \ldots, x^{k-1}e^{\overline{r_k}x}\}$$

to the fundamental set of solutions, where $\overline{r_k} = \lambda_k - i\omega_k$ is the complex conjugate.

Example: Let

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} - 13\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 36y = 0.$$

Then the characteristic polynomial

$$r^4 - 13r^2 + 36r = 0$$

decomposes into linear factors as$(r-2)(r+2)(r-3)(r+3)$, hence we are dealing with the first case above and a fundamental set of solutions is given as $\{e^{-2x}, e^{2x}, e^{-3x}, e^{3x}\}$.