CHAPTER 18 - REMINDER SHEET

The subject of Chapter 18 are solutions to so called Euler equations, i.e. differential equations of the form

$$a_0x^n\frac{\mathrm{d}^ny}{\mathrm{d}x^n}+a_1x^{n-1}\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}+\ldots+a_{n-1}x\frac{\mathrm{d}y}{\mathrm{d}x}+a_ny=0.$$

where $a_i\in\mathbb{R}$ are constants such that $a_0\neq 0$.

The basic ideas to find the fundamental solutions over the interval $x\in(0,\infty)$ are very similar to the ideas in Chapter 17. The main difference is that we don't try to factor the characteristic polynomial but the so called *inicial equation*. We follow the following recipe:

(1) Assume that the solution is of the form $y=x^r$ and plug this into the differential equation. The result is the inicial equation, a polynomial equation in the variable r of degree n.

(2) Solve this equation for the roots $r_1,\ldots,r_k$ and their multiplicities.

(3) If a root r is real root with multiplicity m, then the set of fundamental solutions corresponding to this root are given by

$$\{x^r,x^r\ln|x|,x^r(\ln|x|)^2,\ldots,x^r(\ln|x|)^{m-1}\}.$$

(4) If a root $r=\lambda+i\omega$ is complex root with multiplicity m, then the complex conjugate $\overline{r}=\lambda-i\omega$ is another root of multiplicity m and we get the set of fundamental solutions

$$\{x^r,x^r\ln|x|,x^r(\ln|x|)^2,\ldots,x^r(\ln|x|)^{m-1},x^{\overline{r}},x^{\overline{r}}\ln|x|,x^{\overline{r}}(\ln|x|)^2,\ldots,x^{\overline{r}}(\ln|x|)^{m-1}\}.$$

Example: Let

$$x^3\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-6x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+19x\frac{\mathrm{d}y}{\mathrm{d}x}-27y=0.$$

Then the inicial equation is given by

$$x^3r(r-1)(r-2)x^{r-3}-6x^2r(r-1)x^{r-2}+19xrx^{r-1}-27x^r=0,$$

which we compute to be $r^3-9r^2+27r-27=0$. This decomposes as $(r-3)^3=0$, hence the set of fundamental solutions is given by $\{x^3,x^3\ln|x|,x^3(\ln|x|)^2\}$.