CHAPTER 22 - REMINDER SHEET

The subject of Chapter 22 is the method of variation of parameters. It is an extension of the reduction of order method introduced in Chapter 12. Assume that we have a linear, second-order differential equation in standard form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + P(x)\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)y = g(x),$$

and assume that we know the solution to the associated homogeneous equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + P(x)\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)y = 0.$$

We write this solution in the form $y_h(x) = C_1 y_1(x) + C_2 y_2(x)$. Then we can use the Ansatz

$$y(x) = u(x)y_1(x) + v(x)y_2(x)$$

where the unknown fuunctions $u(x)$ and $v(x)$ replace the constants C_1 and C_2 in the solution of the homogeneouus equation. These fuuntions $u(x)$ and $v(x)$ are called the *varying parameters.*
The system of equations that needs to be solved is

$$\left\{ \begin{array}{l} u'y_1 + v'y_2 = 0, \\ u'y_1' + v'y_2' = g(x) \end{array} \right\} \tag{1}$$

As the associated Wronskian matrix is invertible, this is just a problem of linear algebra. We can invert the matrix to find equations for u' and v' which are readily integrated to give solutions of the original equation.

Example: We want to solve $y''(x) + y(x) = \tan(x)$. The homogeneous solution is given by $y_h(x) = C_1\cos(x) + C_2\sin(x)$. Then we find the equations

$$\left\{ \begin{array}{l} u'\cos(x) + v'\sin(x) = 0, \\ u'(-\sin(x)) + v'\cos(x) = \tan(x), \end{array} \right\} \tag{2}$$

which we can solve for $u'(x) = -\sin(x)\tan(x)$ and $v'(x) = \sin(x)$. Integration of these two functions gives the general solution

$$y = C_1\cos(x) + C_2\sin(x) - \cos(x)\ln(|\sec(x) + \tan(x)|).$$