CHAPTER 24 - REMINDER SHEET

The subject of Chapter 24 is the relationship between Laplace transforms of differential equations and the original equations.

Theorem: If $f(t), f'(t), f^{(2)}(t), \ldots, f^{(n-1)}(t)$ are piecewise smooth on $[0,\infty)$ and of exponential order (i.e. $\lim_{s\to\infty} \mathscr{L}\{f(t)\}(s) = 0$), and if $f^{(n)}(t)$ is piecewise continuous on $[0,\infty)$, then

$$\mathscr{L}\{f^{(n)}(t)\}(s) = s^n \mathscr{L}\{f(t)\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Hence the Laplace transform changes differentiation in terms of t into multiplication by s, thus transforming a differential equation into an algebraic equation.

Example: Take the differential equation $y'' + y = 0$. Then taking the Laplace transform of both sides we obtain

$$\mathscr{L}\{y'' + y\}(s) = \mathscr{L}\{0\}(s),$$

which yields by the above theorem

$$s^2\mathscr{L}\{y\}(s) - sy(0) - y'(0) + \mathscr{L}\{y\}(s) = 0.$$

Given initial conditions $y(0) = a$ and $y'(0) = b$ this gives an algebraic equation in $\mathscr{L}\{y\}(s)$, which we can solve for the Laplace transform of the solution y.

Another important property of the Laplace transform is the *differentiation property*:

$$\frac{\mathrm{d}^n \mathscr{L}\{y\}(s)}{\mathrm{d}s^n} = \mathscr{L}\{y\}^{(n)}(s) = (-1)^n \mathscr{L}\{t^n y\}(s).$$

Example: We want to compute the Laplace transform of $t\exp(e^{kt})$. We know that $\mathscr{L}\{e^{kt}\}(s) = \frac{1}{s-k}$. Then by the differentiation property we have

$$\mathscr{L}\{te^{kt}\}(s) = -\mathscr{L}\{e^{kt}\}'(s) = -\frac{\mathrm{d}}{\mathrm{d}s}(\frac{1}{s-k}) = \frac{1}{(s-k)^2}.$$