

CHAPTER 7 - REMINDER SHEET

The subject of Chapter 7 are first order differential equations in *exact form*. We say that an equation is in exact form if and only if

- (1) The differential equation is written in the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0.$$

- (2) There is a differentiable function $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial \phi}{\partial y} = N(x, y).$$

It is not always straightforward to find ϕ or even to know that it exists. If the equation

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

holds, then there exists a ϕ over a certain subregion of the xy -plane. A basic procedure to find a ϕ is given by the following steps:

- (a) Write

$$\frac{\partial \phi}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial \phi}{\partial y} = N(x, y)$$

and integrate the left equation with respect to x , treating y as a constant, i.e.

$$\int \frac{\partial \phi}{\partial x} dx = \int M(x, y) dx.$$

When integrating the right hand side interpret the constant of integration as an unknown function $h(y)$ in y .

- (b) Substitute the formula for ϕ into the equation

$$\frac{\partial \phi}{\partial y} = N(x, y).$$

This will result in an equation for the differential $h'(y)$. If this equation depends on x , then the differential equation was not exact to begin with, thus we cannot solve it with this method.

- (c) If the formula for $h'(y)$ does not depend on x , then we can proceed and integrate the equation to yield h and thus determine ϕ .

If we have a function $\phi(x, y)$ as described above, then we can rewrite the differential equation as

$$\frac{d}{dx}(\phi) = 0,$$

hence the solution is implicitly given as $\phi(x, y) = 0$.

Just like how linear equation can be easier to solve after multiplying with an integrating factor there are also often situations where multiplying by an integrating factor allows us to write a differential equation in exact form.