Series Tests for Convergence/Divergence

Test	Suppositions	Conclusion	
Test for Divergence If you can see that $\lim_{n\to\infty} a_n \neq 0$, use Test for Divergence		If $lim_{n\to\infty}\, a_n$ does not exist or if $lim_{n\to\infty}\, a_n \neq 0$	$\sum_{n=1}^{\infty} a_n$ diverges
Integral Test Use Integral Test if $\int_1^{\infty} f(x)\,dx$ is easily evaluated.	Suppose function $f(x)$ is • continuous on $[k, \infty)$ • positive on $[k, \infty)$ • ultimately decreasing *($f'(x) < 0$ for $x > N$)* • $a_n = f(n)$	If $\int_k^{\infty} f(x)\,dx$ converges	$\sum_{n=k}^{\infty} a_n$ converges
		If $\int_k^{\infty} f(x)\,dx$ diverges	$\sum_{n=k}^{\infty} a_n$ diverges
Comparison Test Consider using a comparison (or limit comparison) test for series that have form similar to a *p*-series or a geometric series. Rational functions should be compared with *p*-series.	Suppose • $\sum a_n$ and $\sum b_n$ have positive terms for all $n \geq N$ • *Note: $\sum b_n$ is a known series, usually the p-series or geometric series.*	If $\sum b_n$ converges and $a_n \leq b_n\ \forall\, n \geq N$	$\sum a_n$ converges
		If $\sum b_n$ diverges and $a_n \geq b_n\ \forall\, n \geq N$	$\sum a_n$ diverges
Limit Comparison Test Note: The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply the Comparison Test to $\sum \lvert a_n \rvert$ and test for absolute convergence.	Suppose • $\sum a_n$ and $\sum b_n$ have positive terms for all $n \geq N$. • *Note: $\sum b_n$ is a known series, usually the p-series or geometric series.*	If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ *(where c is finite and $c > 0$)*	both series converge or both series diverge
		If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges	$\sum a_n$ converges
		If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges	$\sum a_n$ diverges
Ratio Test The Ratio Test is a convenient test for series that involve factorials or other products (including a constant raised to the nth power).	Suppose • $\sum a_n$ has positive terms	(i) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$	$\sum a_n$ is absolutely convergent
		(ii) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L > 1$	$\sum a_n$ diverges
		(iii) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$	test is inconclusive
Root Test	Suppose • $\sum a_n$ has positive terms for $n \geq N$	(i) If $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho < 1$	$\sum a_n$ converges
		(ii) If $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho > 1$	$\sum a_n$ diverges
		(iii) If $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$	test is inconclusive
Alternating Series Test Use for series of the form $\sum(-1)^{n-1}b_n$.	Given an alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n \quad (b_n > 0)$	If (i) $b_{n+1} \leq b_n$ (ii) $\lim_{n\to\infty} b_n = 0$	$\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges

Definitions	A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum \lvert a_n \rvert$, converges.
	A series that converges but does not converge absolutely **converges conditionally**.
Absolute Convergence Theorem	If $\sum \lvert a_n \rvert$ converges, then $\sum a_n$ converges.

Theorem 1 – Limit Laws

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule*: $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule*: $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Product Rule*: $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
4. *Constant Multiple Rule*: $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (any number k)
5. *Quotient Rule*: $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{A}{B}$ if $B \neq 0$

LIMITS

Theorem 5 – Commonly Occurring Limits

The following six sequences converge to the limits listed below:

1	$\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2	$\lim_{n\to\infty} \sqrt[n]{n} = 1$
3	$\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$
4	$\lim_{n\to\infty} x^n = 0 \quad (\lvert x\rvert < 1)$
5	$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad$ (any x)
6	$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad$ (any x)

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

Theorem 2 – Sandwich Theorem for Sequences

Let $\{a_n\}, \{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all $n \geq N$, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Theorem 3 – Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Theorem 3 states that applying a continuous function to a convergent sequence produces a convergent sequence.

Theorem 4

Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L$$

Theorem 4 enables us to use l'Hopital's Rule to find the limit of some sequences.

Known Series

The **geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r|<1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geq 1$, the geometric series is divergent.

The ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent if $p > 1$ and divergent if $p \leq 1$.

The **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

Telescoping series

Series Definitions / Theorem

Definitions

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$\vdots$$
$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and the **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Theorem 7: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

(Note: The converse of this theorem is not true in general. If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\sum_{n=1}^{\infty} a_n$ is convergent. The harmonic series is an example.)