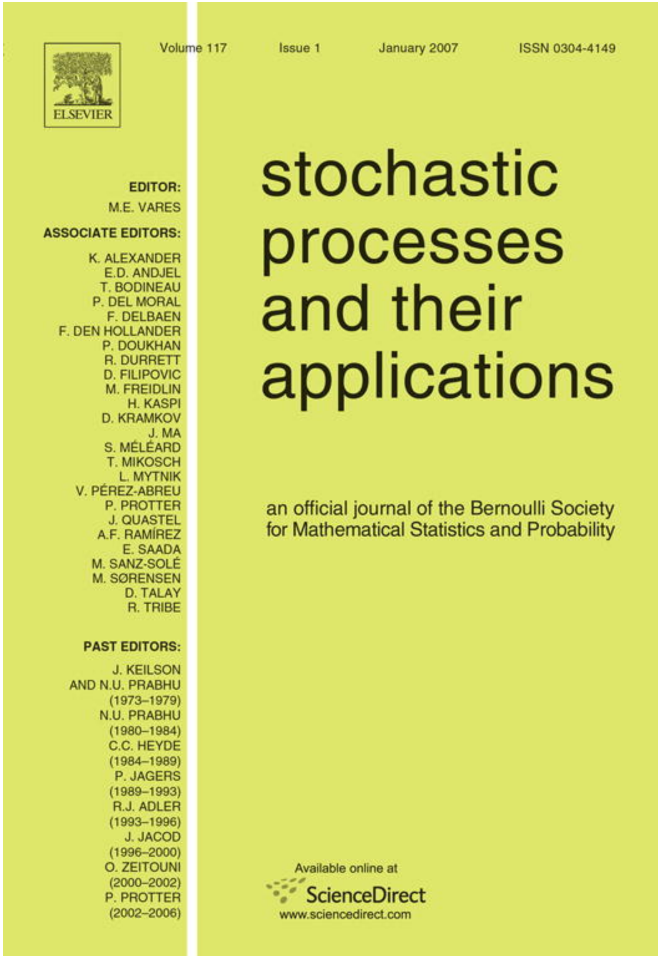
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Approximations and limit theory for quadratic forms of linear processes

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Abstract

The paper develops a limit theory for the quadratic form $Q_{n,X}$ in linear random variables $X_1, \ldots, X_n$ which can be used to derive the asymptotic normality of various semiparametric, kernel, window and other estimators converging at a rate which is not necessarily $n^{1/2}$. The theory covers practically all forms of linear serial dependence including long, short and negative memory, and provides conditions which can be readily verified thus eliminating the need to develop technical arguments for special cases. This is accomplished by establishing a general CLT for $Q_{n,X}$ with normalization $(\mathrm{Var}[Q_{n,X}])^{1/2}$ assuming only $2+\delta$ finite moments. Previous results for forms in dependent variables allowed only normalization with $n^{1/2}$ and required at least four finite moments. Our technique uses approximations of $Q_{n,X}$ by a form $Q_{n,Z}$ in i.i.d. errors $Z_1, \ldots, Z_n$. We develop sharp bounds for these approximations which in some cases are faster by the factor $n^{1/2}$ compared to the existing results.

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1. Introduction

It is well-known that a quadratic form

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$$Q_{n,X} = \sum_{k,t=1}^{n} d_n(k-t) X_k X_t \tag{1.1}$$

in independent identically distributed (i.i.d.) random variables X_k satisfies the central limit theorem (CLT) under unrestrictive assumptions, see [26,19,16,24]. In particular, the CLT holds with the normalization $(\mathrm{Var}(Q_{n,X}))^{1/2}$ which might be different from $n^{1/2}$, and, if the diagonal vanishes, only the second moment of the X_k is needed. Much weaker results are available if the X_k are dependent: the CLT is proved only with normalization $n^{1/2}$, at least four moments of the X_k are assumed and the kernel d_n does not depend on n, see [13,4,14,15], and references therein. Such assumptions are not sufficient in a number of important statistical applications, as discussed in Section 3.

In the present paper we show that the asymptotic results valid for i.i.d. variables extend to dependent linear variables. We approximate $Q_{n,X}$ by a quadratic form

$$Q_{n,Z} = \sum_{k,t=1}^{n} e_n(k-t) Z_k Z_t, \tag{1.2}$$

in i.i.d. variables Z_t which are the innovations of the process $\{X_t\}$ (precise definitions are given in Section 2). This method has been used by several authors, but, assuming at least four moments, the existing research provides only the bound $\mathrm{Var}(Q_{n,X} - Q_{n,Z}) = o(n)$ which leads to the CLT with normalization $\sqrt{n}$. Our Theorem 2.1 provides a much sharper bound which implies that $Q_{n,X} - Q_{n,Z}$ is dominated by $Q_{n,Z}$, no matter how slowly $\mathrm{Var}(Q_{n,Z})$ diverges to ∞. Combined with a new CLT for $Q_{n,Z}$ with a nonvanishing diagonal, established in [8], this approximation leads to the general CLT for $Q_{n,X}$ with normalization $(\mathrm{Var}(Q_{n,X}))^{1/2}$.

The CLT with a general normalization allows us to establish in a unified framework the asymptotic normality of a number of statistics that can be written as quadratic forms $Q_{n,X}$ and depend on a "bandwidth parameter". Such statistics converge at a rate slower than $n^{1/2}$. Important applications include spectral kernel and semiparametric estimation. As a rule, complex technical arguments are needed to establish the asymptotic normality in such cases. While the present paper is obviously also very technical, it aims at providing general results which can be directly applied to avoid problem specific technical work.

Definitions, assumptions and main results are presented in Section 2 which also contains short proofs linking the main results with the detailed theory developed in the following sections. The relevance of the assumptions and findings is discussed in Section 3 which compares in greater detail our results to previous research. It also explains the assumptions and the results and shows how they are motivated by applications to statistics. Section 4 deals with the CLT for general quadratic forms in i.i.d. variables Z_k. The approximation of $Q_{n,X}$ by $Q_{n,Z}$ is developed in Section 5. Section 6 contains several technical lemmas which form the backbone of the proofs in Section 5.

2. Assumptions and main results

We assume that $X_t, t = 1, 2, \ldots, n$, is a realization of a linear process

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \tag{2.1}$$

where $\{Z_t\}$ is a sequence of independent identically distributed random variables with

$$EZ_t = 0, \qquad EZ_t^2 = 1 \tag{2.2}$$

and the coefficients ψ_j satisfy the assumption $\sum_{j=0}^{\infty} \psi_j^2 < \infty$. The spectral density function $f(\lambda)$, $|\lambda| \le \pi$, of $\{X_t\}$ can be written as $f(\lambda) = (2\pi)^{-1} |\Psi(\lambda)|^2$ where $\Psi(\lambda) = \sum_{j=0}^{\infty} \psi_j \mathrm{e}^{-\mathrm{i}\lambda j}$ is the transfer function.

We denote by $\{\eta_n(\lambda), n \ge 1\}$ a family of even real functions, and for integer t define

$$d_n(t) = \int_{-\pi}^{\pi} \eta_n(\lambda)\mathrm{e}^{\mathrm{i}\lambda t}\mathrm{d}\lambda, \qquad e_n(t) = 2\pi \int_{-\pi}^{\pi} \eta_n(\lambda) f(\lambda)\mathrm{e}^{\mathrm{i}\lambda t}\mathrm{d}\lambda.$$

Our first objective is to obtain conditions under which the quadratic form

$$Q_{n,X} = \sum_{k,t=1}^{n} d_n(k-t)X_k X_t \equiv (2\pi n) \int_{-\pi}^{\pi} \eta_n(\lambda) I_n(\lambda)\mathrm{d}\lambda \tag{2.3}$$

of variables $\{X_t\}$ can be approximated by a quadratic form

$$Q_{n,Z} = \sum_{k,t=1}^{n} e_n(k-t)Z_k Z_t \equiv (2\pi n) \int_{-\pi}^{\pi} \eta_n(\lambda) 2\pi f(\lambda) I_{n,Z}(\lambda)\mathrm{d}\lambda \tag{2.4}$$

of i.i.d. noise $\{Z_t\}$, where

$$I_n(\lambda) = \frac{1}{2\pi n} \left| \sum_{j=1}^{n} X_j \mathrm{e}^{\mathrm{i}\lambda j} \right|^2, \qquad I_{n,Z}(\lambda) = \frac{1}{2\pi n} \left| \sum_{j=1}^{n} Z_j \mathrm{e}^{\mathrm{i}\lambda j} \right|^2$$

are the periodograms of the sequence $\{X_t\}$ and of the noise $\{Z_t\}$.

The approximation is based on Barlett's [6] decomposition

$$I_n(\lambda) = 2\pi f(\lambda) I_{n,Z}(\lambda) + (2\pi)^{-1} L_n(\lambda) \tag{2.5}$$

into the main term $2\pi f(\lambda) I_{n,Z}(\lambda) = |\Psi(\lambda)|^2 I_{n,Z}(\lambda)$ and the reminder

$$L_n(\lambda) = 2\pi \left(I_n(\lambda) - |\Psi(\lambda)|^2 I_{n,Z}(\lambda) \right). \tag{2.6}$$

To derive upper bounds on the order of $\mathrm{Var}(Q_{n,X} - Q_{n,Z})$ and $E|Q_{n,X} - Q_{n,Z}|$, we impose the following assumptions.

Assumption 2.1. There exists $d \in (-1/2, 1/2)$ such that ψ_j satisfy

$$\psi_j = O(j^{-1+d}), \qquad |\psi_j - \psi_{j+1}| = O(j^{-2+d}), \qquad \text{if } d \ne 0 \tag{2.7}$$

and, in addition, $\sum_{j=0}^{\infty} \psi_j = 0$, if $d < 0$.

If $d = 0$, then there exists $\alpha > 1$ such that

$$\sum_{j=n}^{\infty} |\psi_j| = O(n^{-\alpha}). \tag{2.8}$$

Assumption 2.2. $\{\eta_n, n = 1, 2, 3, \ldots\}$ is a family of even real functions on $[-\pi, \pi]$ such that for some $-1 < \beta < 1$ and a sequence of constants $k_n \geq 0$,

$$|\eta_n(\lambda)| \leq k_n |\lambda|^{-\beta}, \quad \lambda \in [-\pi, \pi], n \geq 1. \tag{2.9}$$

The following approximation result is a direct consequence of Lemma 5.1.

Theorem 2.1. *Suppose that Assumptions 2.1 and 2.2 hold and assume that*

$$\delta := 2d + \beta < 1/2. \tag{2.10}$$

If the noise $\{Z_t\}$ has finite fourth moment, then

$$[\mathrm{Var}(Q_{n,X} - Q_{n,Z})]^{1/2} = O(r_n(d, \delta)) \tag{2.11}$$

where

$$r_n(d, \delta) = \begin{cases} k_n, & \text{if } d = 0, \\ k_n n^{\max(\delta,0)}, & \text{if } \delta \neq 0 \text{ and } d \neq 0, \\ k_n n^{\max(\delta,0)} \log n, & \text{if } \delta = 0 \text{ and } d \neq 0. \end{cases} \tag{2.12}$$

If the noise $\{Z_t\}$ has the finite second moment, then

$$E|Q_{n,X} - Q_{n,Z}| = O(\bar{r}_n(d, \delta)) \tag{2.13}$$

where

$$\bar{r}_n(d, \delta) = \begin{cases} k_n, & \text{if } d = 0, \\ k_n n^{\max(\delta,d,0)}(1 + \bar{h}_n), & \text{if } d \neq 0 \end{cases} \tag{2.14}$$

and

$$\bar{h}_n = \begin{cases} \log n, & \text{if (a) } \delta = 0 \text{ and } d < 0, \text{ or (b) } \delta = d \text{ and } d > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(Note that $r_n(d, \delta) \leq \bar{r}_n(d, \delta)$.)

The novelty of Theorem 2.1 lies in the sharp upper bounds in terms of δ (2.10), an exponent which reflects the interplay of the rates of growth, as $\lambda \to 0$, of the spectral density $f(\lambda)$ and the functions $\eta_n(\lambda)$. The constants k_n, coming directly from the assumption $|\eta_n(\lambda)| \leq k_n|\lambda|^{-\beta}$, play a secondary role, and were introduced merely to provide a convenient formulation suitable for statistical applications, see Section 3.

To derive the CLT for the quadratic form $Q_{n,X}$ we have to assume essentially only that the main term $Q_{n,Z}$ dominates the approximation error. To state this assumption, introduce the matrix $E_n = (e_n(t-k))_{t,k=1,\ldots,n}$ and denote by $\|E_n\| = (\sum_{t,k=1}^{n} e_n^2(t-k))^{1/2}$ its Euclidean norm. In the following, for $a_n, b_n \geq 0$, $a_n \asymp b_n$ means that $C_1 b_n \leq a_n \leq C_2 b_n$, $n \geq 1$, for some $C_1, C_2 > 0$.

Theorem 2.2. *Suppose that Assumptions 2.1 and 2.2 and condition (2.10) hold. Assume that $EZ_t^4 < \infty$ and*

$$\frac{r_n(d, \delta)}{\|E_n\|} \to 0. \tag{2.15}$$

Then, as $n \to \infty$,

$$\mathrm{Var}(Q_{n,X})/\mathrm{Var}(Q_{n,Z}) \to 1, \qquad \mathrm{Var}(Q_{n,X}) \asymp \|E_n\|^2 \tag{2.16}$$

and

$$(\mathrm{Var}(Q_{n,X}))^{-1/2}(Q_{n,X} - EQ_{n,X}) \xrightarrow{d} N(0,1). \tag{2.17}$$

Proof of Theorem 2.2. Bound (2.11) and condition (2.15) imply the first relation of (2.16). Since

$$\mathrm{Var}(Q_{n,Z}) = 2\sum_{t,k=1:t\neq k}^{n} e_n^2(t-k) + \mathrm{Var}(Z_0^2)e_n^2(0)n, \tag{2.18}$$

$$\mathrm{Var}(Q_{n,Z}) \asymp \|E_n\|^2. \tag{2.19}$$

In view of (2.16) and (2.15), the convergence

$$\frac{Q_{n,X} - EQ_{n,X}}{(\mathrm{Var}(Q_{n,X}))^{1/2}} = \frac{Q_{n,Z} - EQ_{n,Z}}{(\mathrm{Var}(Q_{n,Z}))^{1/2}} + o_P(1) \xrightarrow{d} N(0,1) \tag{2.20}$$

follows from Theorem 4.2 below. ■

Statistical applications involve the integrated periodograms

$$T_{n,X} = \int_{-\pi}^{\pi} \eta_n(\lambda) I_n(\lambda)\mathrm{d}\lambda = (2\pi n)^{-1} Q_{n,X},$$

$$T_{n,Z} = 2\pi \int_{-\pi}^{\pi} \eta_n(\lambda) f(\lambda) I_{n,Z}(\lambda)\mathrm{d}\lambda = (2\pi n)^{-1} Q_{n,Z}.$$

Even though these two forms are just multiples of $Q_{n,X}$ and $Q_{n,Z}$, in the remainder of this section we formulate several results in terms of $T_{n,X}$ and $T_{n,Z}$ because such formulations are more usual and convenient in statistics.

For ease of reference, we begin with Remark 2.1 which restates Theorem 2.1.

Remark 2.1. If Assumptions 2.1 and 2.2 and condition (2.10) hold, then

(i) if $EZ_t^4 < \infty$, then $[\mathrm{Var}(T_{n,X} - T_{n,Z})]^{1/2} = O(n^{-1} r_n(d,\delta))$; (2.21)

(ii) if $EZ_t^2 < \infty$, then $E|T_{n,X} - T_{n,Z}| = O(n^{-1}\bar{r}_n(d,\delta))$. (2.22)

Corollary 2.1 focuses on conditions under which the centering constants $ET_{n,X}$ in the CLT can be replaced by the more convenient and explicit constants $ET_{n,Z} = \int_{-\pi}^{\pi} \eta_n(\lambda) f(\lambda)\mathrm{d}\lambda$.

Corollary 2.1. *Suppose that Assumptions 2.1 and 2.2 and condition (2.10) are satisfied.*

(i) *If* $EZ_t^4 < \infty$ *and (2.15) hold, then*

$$[\mathrm{Var}(T_{n,X})]^{-1/2}(T_{n,X} - ET_{n,X}) \xrightarrow{d} N(0,1). \tag{2.23}$$

(ii) *If* $EZ_t^4 < \infty$ *and*

$$\frac{\bar{r}_n(d,\delta)}{\|E_n\|} \to 0, \tag{2.24}$$

hold, then

$$[\mathrm{Var}(T_{n,X})]^{-1/2}\left(T_{n,X}-\int_{-\pi}^{\pi}\eta_n(\lambda)f(\lambda)\mathrm{d}\lambda\right)\xrightarrow{d}N(0,1). \tag{2.25}$$

(Note that $\bar{r}_n(d,\delta)\le k_n n^{\max(d,\delta,0)}\log n$.)

(iii) *If (2.24) holds and either condition*

$$EZ_t^2<\infty \quad \textit{and} \quad \int_{-\pi}^{\pi}\eta_n(\lambda)f(\lambda)\mathrm{d}\lambda\equiv 0 \tag{2.26}$$

or condition

$$EZ_t^{2+\delta}<\infty \quad \textit{for some } \delta>0 \quad \textit{and} \quad \int_{-\pi}^{\pi}\eta_n(\lambda)f(\lambda)\mathrm{d}\lambda=o(n^{-1/2}\|E_n\|) \tag{2.27}$$

is satisfied, then

$$\mathrm{Var}(T_{n,X})=\frac{\|E_n\|^2}{2(\pi n)^2}(1+o(1)) \tag{2.28}$$

and

$$\frac{\sqrt{2}\pi n}{\|E_n\|}\left(T_{n,X}-\int_{-\pi}^{\pi}\eta_n(\lambda)f(\lambda)\mathrm{d}\lambda\right)\xrightarrow{d}N(0,1). \tag{2.29}$$

Proof of Corollary 2.1. Convergence (2.23) is equivalent to (2.17).

We now verify (2.25). In (6.3) of Lemma 6.1 we show that under Assumption 2.1 the spectral density of $\{X_t\}$ satisfies

$$f(\lambda)\le C|\lambda|^{-2d},\quad \lambda\in[-\pi,\pi]. \tag{2.30}$$

Therefore the function $g_n(\lambda)=\eta_n(\lambda)f(\lambda)$ satisfies condition (4.7) with $\alpha=\delta$. Assumption (2.24) implies condition (4.9) of Theorem 4.2. Since $ET_{n,Z}=\int_{-\pi}^{\pi}\eta_n(\lambda)f(\lambda)\mathrm{d}\lambda$, we conclude that

$$[\mathrm{Var}(T_{n,Z})]^{-1/2}\left(T_{n,Z}-\int_{-\pi}^{\pi}\eta_n(\lambda)f(\lambda)\mathrm{d}\lambda\right)\xrightarrow{d}N(0,1). \tag{2.31}$$

By (2.16), it suffices to show that

$$n\|E_n\|^{-1}E|T_{n,Z}-T_{n,X}|\to 0. \tag{2.32}$$

Since assumption (2.24) implies $\bar{r}_n(d,\delta)/\|E_n\|\to 0$, (2.32) follows from (2.22).

Part (iii) follows in a similar manner from part (iii) of Theorem 4.1. ■

Relation (2.21) leads to a sharp bound on the L^2 norm of $T_{n,X}$ with the random approximation $T_{n,Z}$. Corollary 2.2 establishes corresponding bounds with deterministic centering constants. Even though these bounds are weaker, deterministic centering is required in most statistical applications.

Corollary 2.2. *Suppose that Assumptions 2.1 and 2.2 and conditions (2.10) and $EZ_t^4<\infty$ hold. Then*

$$(E|T_{n,X}-ET_{n,X}|^2)^{1/2}\le Ck_n n^{-1/2}, \tag{2.33}$$

and

$$\left(E\left|T_{n,X}-\int_{-\pi}^{\pi}\eta_n(\lambda)f(\lambda)\mathrm{d}\lambda\right|^2\right)^{1/2}\leq Ck_n n^{-1/2} \tag{2.34}$$

where $C>0$ *does not depend on* n *and* k_n*.*

Proof of Corollary 2.2. By (2.21) and (2.19)

$$E[T_{n,X}-ET_{n,X}]^2\leq Cn^{-2}[r_n^2(d,\delta)+\|E_n\|^2].$$

By Parseval's equality,

$$\begin{aligned}\|E_n\|^2&\leq Cn\sum_{t=-\infty}^{\infty}e_n^2(t)\leq Cn\int_{-\pi}^{\pi}|\eta_n(\lambda)f(\lambda)|^2\mathrm{d}\lambda\\&\leq Ck_n^2 n\int_{-\pi}^{\pi}|\lambda|^{-2\delta}\mathrm{d}\lambda\leq Ck_n^2 n.\end{aligned} \tag{2.35}$$

Since $r_n^2(d,\delta)\leq Ck_n^2 n$, (2.33) follows. Bound (2.34) follows from (2.33), (2.22) and (2.14). ■

Remark 2.2. To lighten the notation, we assume that $EZ_t^2=1$. Theorems 2.1 and 2.2 and Corollaries 2.1 and 2.2 remain valid, without any modification in formulation, if this assumption is replaced by $EZ_t^2=\sigma^2>0$.

3. Discussion of the results of Section 2

In Assumption 2.1 we impose only an upper bound on the rate of the ψ_j, so it holds obviously if the coefficients ψ_j satisfy the following stronger condition in which the asymptotic rate is exactly specified:

Assumption 3.1. There exists $d\in(-1/2,1/2)$ and a constant $c\neq 0$ such that

$$\psi_j=\begin{cases}cj^{-1+d}(1+O(j^{-1})), & \text{if } d\in[0,1/2),\\ cj^{-1+d}(1+O(j^{-1}))\text{ and }\sum_{j=0}^{\infty}\psi_j=0, & \text{if } d\in(-1/2,0).\end{cases} \tag{3.1}$$

If $d=0$, then (2.8) holds.

Assumption 3.1 is motivated by commonly used time series models and often simplifies technical arguments.

Assumption 3.1 and a standard argument, see Theorem 2.15 in [27], imply that the spectral density $f(\lambda)$ satisfies

$$f(\lambda)=c|\lambda|^{-2d}(1+o(1)),\quad \text{as }\lambda\to 0, \tag{3.2}$$

with some $c>0$, whereas Assumption 2.1 implies, in general, merely $f(\lambda)=O(|\lambda|^{-2d})$.

Assumption 2.2 on the Fourier transform $\eta_n(\lambda)$ of the weights $d_n(k)$ is weaker than the usual conditions in that it allows the function η_n to depend on n. Such a relaxation is needed in a number of statistical problems including semiparametric and kernel estimation.

If a parametric model is correctly specified, the asymptotic normality with rate $n^{1/2}$ can be derived using an appropriate CLT for a quadratic form in which η does not depend on n. In practice, however, a model can rarely be correctly specified and it must be fitted to the data before estimation. The fitting procedure will typically yield a different model for different values of n. The function η_n will then depend on n though an "order parameter" and the rate of convergence of relevant statistics will be slower than $n^{1/2}$. For example, if an autoregressive model of order p is fitted, the asymptotic theory requires that $p = p_n$ be a function of n. Spectral estimators then converge at the rate $(n/p_n)^{1/2}$, see e.g. [7,9]. At the end of this section we include a more detailed example related to spectral density estimation.

Connections between approximation results similar to those in Theorem 2.1 and asymptotic inference for time series go back to the work of [18] and [17]. They have been instrumental in establishing asymptotic properties of linear processes and rely on the identity

$$Q_{n,X} - Q_{n,Z} = n \int_{-\pi}^{\pi} L_n(\lambda)\eta_n(\lambda)\mathrm{d}\lambda. \tag{3.3}$$

Relying on earlier results, [10], Proposition 10.8.5 on p. 387, show that for an invertible causal ARMA process $\{X_t\}$ and any continuous even function $\eta(\lambda)$ on $[-\pi, \pi]$,

$$E|Q_{n,X} - Q_{n,Z}| = nE\left|\int_{-\pi}^{\pi} L_n(\lambda)\eta(\lambda)\mathrm{d}\lambda\right| = o(n^{1/2}). \tag{3.4}$$

Relation (3.4) is used to establish the asymptotic normality of several estimators of the parameters of ARMA processes. This approach was extended to more general parametric models by Mikosch et al. [25], Kokoszka and Taqqu [22,23] and Kokoszka and Mikosch [21]. All these papers assume that the function $\eta(\lambda)$ does not depend on n.

To compare our bounds and their consequences to these earlier results, suppose first that $\{X_t\}$ is a linear process with bounded spectral density, e.g. an ARMA process. In this case $d = 0$ and assuming that the functions η_n are bounded, i.e. $k_n \equiv K, \beta = 0$, (2.13) implies

$$E|Q_{n,X} - Q_{n,Z}| = O(1), \tag{3.5}$$

which is a much sharper bound than (3.4). Moreover, approximation (3.5) implies that $Q_{n,X}$ satisfies the CLT as long as $\mathrm{Var}(Q_{n,X}) \to \infty$, with no restriction on the variance order. Previous research for linear processes, see [14] and references therein, established the CLT assuming that $\mathrm{Var}(Q_{n,X}) \sim Cn$ and $\eta_n(\lambda) \equiv \eta(\lambda)$ does not depend on n.

Bound (3.5) also hold for $d \neq 0$ provided $EZ_t^4 < \infty$ and $k_n \equiv K, \beta < -2d$, see (2.12). Another important case, relevant to the inference for long memory processes, is $d > 0$ and $k_n \equiv K, \beta = -2d + \varepsilon$, where $\varepsilon > 0$ is arbitrarily small, see [12] and [11]. The right-hand side of (3.5) is then $O(n^{\varepsilon})$ if $EZ_t^4 < \infty$ and $O(n^d)$ if only the second moment $EZ_t^2 < \infty$ is finite. Since $d < 1/2$, both bounds improve the previous bound $o(n^{1/2})$.

We also note that the existing research on the convergence of quadratic forms essentially covers two cases: (1) the X_t have at least finite fourth moment, (2) the X_t are in the domain of attraction of an infinite variance stable law. As far as we know, no results are available if the X_t have infinite fourth moment but finite second moment. The latter case is of both theoretical and practical importance, see e.g. [2,20]. Part (ii) of Corollary 2.1 fills this gap. It shows that for the validity of CLT the finite fourth moment is not needed when the diagonal of E_n vanishes or its Euclidean norm is dominated by $\|E_n\|$.

We conclude this section by illustrating how our theory can strengthen important results in spectral analysis of time series. To focus attention and limit the exposition, assume that $\{X_t\}$

is a linear short memory sequence which satisfies Assumption 2.1 with $d = 0$ and that we are interested in estimating its spectral density at the zero frequency. We estimate $f(0)$ by the estimator

$$\widehat{f}(0) = \int_{-\pi}^{\pi} \eta_n(\lambda) I_n(\lambda) \mathrm{d}\lambda$$

where

$$\eta_n(\lambda) = (2\pi q)^{-1} \left| \sum_{j=1}^{q} \mathrm{e}^{\mathrm{i}j\lambda} \right|^2$$

is the Fejér kernel. The estimator $\widehat{f}(0)$ is the classical lag window estimator with the Bartlett window (see e.g. [10] pp. 354–362). The bandwidth q is the maximum autocovariance lag used in the estimation and to ensure consistency, it must be assumed that $q = q_n$ (and so η_n) is a function of the sample size n such that

$$q \to \infty, \quad q = o(n), \quad \text{as } n \to \infty.$$

Existing results, see e.g. [3], Theorem 9.4.1, show that $(n/q)^{1/2}(\widehat{f}(0) - E\widehat{f}(0))$ converges to a mean zero normal distribution. These results are derived assuming that X_t has at least finite fourth moment. Centering by $E\widehat{f}(0)$ is not convenient in practice because this quantity cannot be computed explicitly, making the analysis of bias difficult.

We will show that our results directly imply that the asymptotic normality holds if only $2+\delta$ moments are assumed and that a simple deterministic centering is possible which leads to an asymptotic upper bound on the bias.

Assume that f is continuous on $[-\pi, \pi]$ and that

$$f(\lambda) = f(0) + O(\lambda^2), \quad \text{as } \lambda \to 0. \tag{3.6}$$

Since $|\eta_n(\lambda)| \leq Cq$, $|\lambda| \leq \pi$, then $\eta_n(\lambda)$ satisfies Assumption 2.2 with $k_n = Cq$ and $\beta = 0$. Moreover, since $d = 0$, $\delta = 2d + \beta = 0$.

To derive the asymptotic distribution of $\widehat{f}(0)$ we use (2.29) of Corollary 2.1. Since f is continuous,

$$\int_{-\pi}^{\pi} \eta_n(\lambda) f(\lambda) \mathrm{d}\lambda = (2\pi q)^{-1} \int_{-\pi}^{\pi} \left| \sum_{j=1}^{q} \mathrm{e}^{\mathrm{i}j\lambda} \right|^2 f(\lambda) \mathrm{d}\lambda \to f(0), \quad \text{as } n \to \infty. \tag{3.7}$$

On the other hand, it is straightforward to check that

$$\begin{aligned} \|E_n\|^2 &= \sum_{t,k=1}^{n} e_n^2(t-k) = \int_{-\pi}^{\pi} \left| \sum_{j=1}^{n} \mathrm{e}^{\mathrm{i}t(x+y)} \right|^2 \eta_n(x) f(x) \eta_n(y) f(y) \mathrm{d}x \mathrm{d}y \\ &\sim qn(8/3)\pi^2 f(0)^2, \end{aligned} \tag{3.8}$$

see e.g. Proposition A.1 of [1]. Since $n^{-1/2}\|E_n\| \sim cq^{1/2} \to \infty$ and (3.7) holds, condition (2.27) of Corollary 2.1 is satisfied. Therefore (2.29) implies that if $EZ_t^{2+\delta} < \infty$ for some $\delta > 0$, then

$$(n/q)^{1/2} \left(\widehat{f}(0) - \int_{-\pi}^{\pi} \eta_n(x) f(x) \mathrm{d}x \right) \xrightarrow{d} N\left(0, \frac{4}{3} f^2(0)\right). \tag{3.9}$$

We note that relation (3.9) does not follow from any existing CLT's for quadratic forms of linear processes because it involves rate of convergence different than $\sqrt{n}$ and the function η_n which depends on n.

Condition (3.6), see Proposition A.2 in [1], leads to the following upper bound on the bias:

$$\int_{-\pi}^{\pi} \eta_n(x) f(x) \mathrm{d}x - f(0) = O(q^{-1}).$$

4. CLT for quadratic forms of i.i.d. random variables

To study the asymptotic behavior of the main term $Q_{n,Z}$, we need a CLT for quadratic forms in i.i.d. variables Z_k with non-zero diagonal elements. For ease of reference, we state here two results established in [8].

Consider the general quadratic form

$$T_n = \sum_{t,k=1}^{n} a_{n;tk} Z_t Z_k$$

where the $a_{n;tk}$ are entries of a real symmetric matrix $A_n = (a_{n;tk})_{t,k=1,\ldots,n}$. Denote by $\|A_n\| = (\sum_{t,k=1}^{n} a_{n;tk}^2)^{1/2}$ the Euclidean norm and by $\|A_n\|_{sp} = \max_{\|x\|=1} \|A_n x\|$ the spectral norm of the matrix A_n.

Theorem 4.1. *Assume that*

$$\frac{\|A_n\|_{sp}}{\|A_n\|} \to 0. \tag{4.1}$$

(i) *If* $EZ_t^4 < \infty$, *then*

$$(\mathrm{Var}(T_n))^{-1/2}(T_n - ET_n) \xrightarrow{d} N(0, 1). \tag{4.2}$$

(ii) *If*

$$EZ_t^2 < \infty \quad \text{and} \quad a_{n;tt}^2 = 0, \quad t = 1, \ldots, n, \tag{4.3}$$

or

$$EZ_t^{2+\delta} < \infty \quad (\text{for some } \delta > 0) \quad \text{and} \quad \sum_{t=1}^{n} a_{n;tt}^2 = o(\|A_n\|^2), \tag{4.4}$$

then

$$\frac{1}{\sqrt{2}\|A_n\|}(T_n - ET_n) \xrightarrow{d} N(0, 1). \tag{4.5}$$

Next we consider the case when A_n is a Toeplitz matrix with entries

$$a_{n;tk} = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(t-k)x} g_n(x) \mathrm{d}x, \quad t, k = 1, \ldots, n, \tag{4.6}$$

where $g_n(x)$, $|x| \le \pi$ is an even real function. Then $\|A_n\|_{sp}$ in (4.1) can be evaluated in terms of the function $g_n(x)$.

Theorem 4.2. *Let A_n be a Toeplitz matrix with entries $a_{n;tk}$ given by (4.6). Assume that there exist $0 \le \alpha < 1$ such that uniformly in $|\lambda| \le \pi$,*

$$|g_n(\lambda)| \le k_n|\lambda|^{-\alpha}, \quad n \ge 1. \tag{4.7}$$

(i) *Then*

$$\|A_n\|_{sp} \le Ck_n n^{\alpha} \quad n \ge 1. \tag{4.8}$$

(ii) *Moreover, if*

$$\frac{k_n n^{\alpha}}{\|A_n\|} \to 0 \tag{4.9}$$

and $EZ_t^4 < \infty$ then the quadratic form T_n satisfies the CLT (4.2), whereas if Z_t and $a_{n;tt}$ satisfy either (4.3) or (4.4) then the CLT (4.5) holds.

5. Approximation lemma

In this section we derive the approximation Lemma 5.1 which is equivalent to Theorem 2.1. The functions $r_n(d, \delta)$ and $\bar{r}_n(d, \delta)$ are defined in (2.12) and (2.14), respectively.

Lemma 5.1. *Assume that the observations X_t follow (2.1) and (2.2). Suppose that Assumptions 2.1 and 2.2 are satisfied and (2.10) holds.*

Then, the remainder term $L_n(\lambda)$ in decomposition (2.5) satisfies

$$\left(E\left|\int_{-\pi}^{\pi}(L_n(\lambda) - EL_n(\lambda))\eta_n(\lambda)\mathrm{d}\lambda\right|^2\right)^{1/2} = O(n^{-1}r_n(d, \delta)), \tag{5.1}$$

if $EZ_t^4 < \infty$, and

$$E\left|\int_{-\pi}^{\pi} L_n(\lambda)\eta_n(\lambda)\mathrm{d}\lambda\right| = O(n^{-1}\bar{r}_n(d, \delta)), \tag{5.2}$$

if $EZ_t^2 < \infty$.

First we prove the following lemma.

Lemma 5.2. *Let $|d| < 1/2$ and $|\beta| < 1$.*

(i) *If $1 + 2d + 2\beta \le 0$, then $\delta = 2d + \beta < 0$ and there exists $1 > \beta' > \beta$ such that*

$$1 + 2d + 2\beta' > 0 \quad \text{and} \quad \delta' = 2d + \beta' < 0. \tag{5.3}$$

(ii) *If $d < 0$ and $d + \beta \le 0$, then $\delta = 2d + \beta < 0$ and there exists $1 > \beta' > \beta$ such that*

$$d + \beta' > 0, \quad 1 + 2d + 2\beta' > 0 \quad \text{and} \quad \delta' = 2d + \beta' < 0. \tag{5.4}$$

In the cases (i) *and* (ii),

$$r_n(d, \delta) = r_n(d, \delta'), \qquad \bar{r}_n(d, \delta) = \bar{r}_n(d, \delta').$$

Proof of Lemma 5.2. (i) Note that $1 + 2d + 2\beta = 1 + 2\delta - 2d \le 0$ implies that $\beta \le -1/2 - d$ and $\delta < 0$ because $d < 1/2$. Then $\beta' = -1/2 - d + \epsilon$ with sufficiently small $\epsilon > 0$ has properties $\beta < \beta' < 1$, $1 + 2d + 2\beta' = 2\epsilon > 0$ and $\delta' = 2d + \beta' = -1/2 + d + \epsilon < 0$.

(ii) If $d < 0$ and $d + \beta \le 0$ then $\delta = 2d + \beta < 0$. Therefore $\beta' = -d + \epsilon$ with sufficiently small $\epsilon > 0$ has properties $\beta < \beta' < 1$, $d + \beta' = \epsilon > 0$ and $\delta' = 2d + \beta' = d + \epsilon < 0$. ∎

Proof of Lemma 5.1. *Proof of* (5.1). In view of Lemma 5.2 we assume in the proof that

$$1 + 2d + 2\beta > 0 \tag{5.5}$$

and

$$d + \beta > 0 \quad \text{if } d < 0, \tag{5.6}$$

otherwise Assumption 2.2 can be weakened replacing β by $\beta' > \beta$, without affecting bound (5.1) of Lemma 5.1. We split the proof into two cases $d \neq 0$ and $d = 0$.

(i) Consider first the case $d \neq 0$, $|d| < 1/2$. Denote by

$$w_X(\lambda) = \frac{1}{(2\pi n)^{1/2}} \sum_{j=1}^{n} X_j e^{-i\lambda j}, \qquad w_Z(\lambda) = \frac{1}{(2\pi n)^{1/2}} \sum_{j=1}^{n} Z_j e^{-i\lambda j}$$

the discrete Fourier transforms of the sequences X_k and Z_k and set

$$\begin{aligned} \Delta_n(\lambda) &= (2\pi n)^{1/2} (w_X(\lambda) - \Psi(\lambda) w_Z(\lambda)) \\ &= \sum_{j=0}^{\infty} \psi_j e^{-i\lambda j} \left(\sum_{k=1-j}^{n-j} Z_k e^{-i\lambda k} - \sum_{k=1}^{n} Z_k e^{-i\lambda k} \right). \end{aligned} \tag{5.7}$$

Then the remainder term $L_n(\lambda) = 2\pi \left(I_n(\lambda) - |\Psi(e^{i\lambda})|^2 I_{n,Z}(\lambda) \right)$ in (2.5) can be written as

$$L_n(\lambda) = n^{-1} \left[|\Delta_n(\lambda)|^2 + 2\Re \left(\Delta_n(\lambda) \Psi(\lambda) \left(\sum_{l=1}^{n} Z_l e^{i\lambda l} \right) \right) \right]. \tag{5.8}$$

It follows from (5.7) that $\Delta_n(\lambda)$ can be represented as a linear process

$$\Delta_n(\lambda) = \sum_{k=-\infty}^{n} e^{-i\lambda k} d_k(\lambda) Z_k - c_n(\lambda) \sum_{k=1}^{n} e^{-i\lambda k} Z_k \tag{5.9}$$

where

$$d_k(\lambda) = \begin{cases} \displaystyle\sum_{j=1-k}^{n-k} \psi_j e^{-i\lambda j}, & \text{for } k \le 0, \\ \displaystyle -\sum_{j=n-k+1}^{n} \psi_j e^{-i\lambda j}, & \text{for } 1 \le k \le n \end{cases}$$

and

$$c_n(\lambda) = \sum_{j=n+1}^{\infty} \psi_j e^{-i\lambda j}.$$

By assumption (2.7), for $k \le 0$, $|d_k(\lambda)| \le Cn(1 + |k|)^{d-1}$, so $\sum_{k=-\infty}^{0} |d_k(\lambda)|^2 < \infty$, and therefore the linear process (5.9) is well defined for any fixed n.

We now introduce the following coefficients:

$$\nu_n(k,t) = \int_{-\pi}^{\pi} e^{i(t-k)\lambda} d_k(\lambda)\overline{d_t(\lambda)}|\eta_n(\lambda)|d\lambda, \tag{5.10}$$

$$\beta_n(k,t) = \int_{-\pi}^{\pi} e^{i(t-k)\lambda} d_k(\lambda)\,\Psi(\lambda)\eta_n(\lambda)d\lambda, \tag{5.11}$$

$$\mu_n(k,t) = \int_{-\pi}^{\pi} e^{i(t-k)\lambda} |c_n(\lambda)|^2|\eta_n(\lambda)|d\lambda, \tag{5.12}$$

$$\zeta_n(k,t) = \int_{-\pi}^{\pi} e^{i(t-k)\lambda} c_n(\lambda)\,\Psi(\lambda)\eta_n(\lambda)d\lambda. \tag{5.13}$$

Set

$$Y_n = \int_{-\pi}^{\pi} |\Delta_n(\lambda)|^2|\eta_n(\lambda)|d\lambda \tag{5.14}$$

and write

$$\int_{-\pi}^{\pi} \Delta_n(\lambda)\,\Psi(\lambda)\left(\sum_{l=1}^{n} Z_l e^{i\lambda l}\right)\eta_n(\lambda)d\lambda =: V_{n,1} - V_{n,2},$$

where

$$\begin{aligned} V_{n,1} &= \int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{n} e^{-i\lambda k} d_k(\lambda) Z_k\right)\left(\sum_{t=1}^{n} Z_t e^{i\lambda t}\right)\Psi(\lambda)\eta_n(\lambda)d\lambda \\ &= \sum_{k=-\infty}^{n}\sum_{t=1}^{n} \beta_n(k,t) Z_k Z_t, \end{aligned} \tag{5.15}$$

$$V_{n,2} = \int_{-\pi}^{\pi}\left|\sum_{k=1}^{n} e^{-i\lambda k} Z_k\right|^2 c_n(\lambda)\,\Psi(\lambda)\eta_n(\lambda)d\lambda = \sum_{k=1}^{n}\sum_{t=1}^{n}\zeta_n(k,t) Z_k Z_t. \tag{5.16}$$

Thus we can estimate:

$$\begin{aligned} &E\left|\int_{-\pi}^{\pi}(L_n(\lambda) - EL_n(\lambda))\eta_n(\lambda)d\lambda\right|^2 \\ &\quad \le Cn^{-2}(E|Y_n|^2 + E|V_{n,1} - EV_{n,1}|^2 + E|V_{n,2} - EV_{n,2}|^2). \end{aligned} \tag{5.17}$$

To prove (5.1) it suffices to show that

$$E|Y_n|^2 = O(r_n^2(d,\delta)) \tag{5.18}$$

and

$$E|V_{n,i} - EV_{n,i}|^2 = O(r_n^2(d,\delta)), \quad i = 1, 2. \tag{5.19}$$

Observing that

$$|\Delta_n(\lambda)|^2 \le C\left(\left|\sum_{k=-\infty}^{-n} e^{-i\lambda k} d_k(\lambda) Z_k\right|^2 + \left|\sum_{k=-n+1}^{0} e^{-i\lambda k} d_k(\lambda) Z_k\right|^2\right.$$

$$+ \left|\sum_{k=1}^{n} e^{-i\lambda k} d_k(\lambda) Z_k\right|^2 + \left|c_n(\lambda)\sum_{k=1}^{n} e^{-i\lambda k} Z_k\right|^2\Bigg)$$

we obtain

$$0 \le Y_n \le C\left(\sum_{k,t=-n+1}^{0} \nu_n(k,t) Z_k Z_t + \sum_{k,t=1}^{n} \nu_n(k,t) Z_k Z_t \right.$$
$$\left. + \sum_{k,t=-\infty}^{-n} \nu_n(k,t) Z_k Z_t + \sum_{k,t=1}^{n} \mu_n(k,t) Z_k Z_t\right).$$

Hence,

$$EY_n \le C\left(\sum_{k=-\infty}^{n} \nu_n(k,k) + \sum_{k=1}^{n} \mu_n(k,k)\right) =: Cm_n. \tag{5.20}$$

Applying elementary estimate

$$E\left(\sum_{k\in K, t\in T} a(k,t)(Z_k Z_t - E[Z_k Z_t])\right)^2 \le C \sum_{k\in K, t\in T} |a(k,t)|^2 \tag{5.21}$$

where C does not depend on K, T and $a(k,t)$, we obtain

$$EY_n^2 \le C\left(\sum_{k,t=-n+1}^{0} \nu_n^2(k,t) + \sum_{k,t=1}^{n} \nu_n^2(k,t) + \sum_{k,t=-\infty}^{-n} \nu_n^2(k,t) + \sum_{k,t=1}^{n} \mu_n^2(k,t) + m_n^2\right)$$
$$=: C\left(s_n^{(1)} + s_n^{(2)} + s_n^{(3)} + s_n^{(4)} + m_n^2\right). \tag{5.22}$$

In addition we shall bound

$$m_n^2 \le C\left(\left(\sum_{k=-n+1}^{0} \nu_n(k,k)\right)^2 + \left(\sum_{k=1}^{n} \nu_n(k,k)\right)^2 + \left(\sum_{k=-\infty}^{-n} \nu_n(k,k)\right)^2\right.$$
$$\left. + \left(\sum_{k=1}^{n} \mu_n(k,k)\right)^2\right) =: C(v_n^{(1)} + v_n^{(2)} + v_n^{(3)} + v_n^{(4)}). \tag{5.23}$$

Thus (5.18) follows if we show that

$$v_n^{(i)} = O(r_n^2(d,\delta)), \qquad s_n^{(i)} = O(r_n^2(d,\delta)), \quad i = 1, \dots, 4. \tag{5.24}$$

Let $d > 0$. By bound (6.7) of Lemma 6.3 below we have that $|\nu_n(k,t)|^2 \le C k_n^2 |k|_+^{-1+\delta} |t|_+^{-1+\delta}$, and therefore

$$s_n^{(1)} \le C k_n^2 \sum_{k,t=-n+1}^{0} |k|_+^{-1+\delta} |t|_+^{-1+\delta} = O(r_n^2(d,\delta)),$$

$$v_n^{(1)} \le C\left(k_n \sum_{k=-n+1}^{0} |k|_+^{-1+\delta}\right)^2 = O(r_n^2(d,\delta)),$$

to prove (5.24) for $i = 1$.

The proof of (5.24) for $s_n^{(i)}$, and $v_n^{(i)}$, $i = 2$ and $i = 4$ follows in a similar fashion using the bounds (6.7) and (6.8) of Lemma 6.3 below.

To estimate $s_n^{(3)}$, and $v_n^{(3)}$ we use bound (6.7.3) of Lemma 6.3 which yields

$$\begin{aligned} s_n^{(3)} &\le Ck_n^2 \sum_{k,t=-\infty}^{-n} n^{2(1+\beta)}|k|^{-2(1-d)}|t|^{-2(1-d)} \\ &= O(k_n^2 n^{4d+2\beta}) = O(k_n^2 n^{2\delta}) = O(r_n^2(d,\delta)) \end{aligned}$$

whereas

$$v_n^{(3)} \le C\left(k_n \sum_{k=-\infty}^{-n} n^{1+\beta}|k|^{-2+2d}\right)^2 = O(k_n^2 n^{2\delta}) = O(r_n^2(d,\delta)).$$

In the case $-1/2 < d < 0$ proof of (5.24) follows exactly the same line as for $d > 0$, using estimates (6.7) and (6.8) of Lemma 6.3.

It remains to show (5.19). We start with the case $i = 1$. By (5.21),

$$E|V_{n,1} - EV_{n,1}|^2 \le \sum_{k=-\infty}^{n} q_n(k), \tag{5.25}$$

where

$$q_n(k) = \sum_{t=1}^{n} |\beta_n(k,t)|^2. \tag{5.26}$$

By Parseval's equality and definition (5.11) of $\beta_n(k,t)$ it follows that for any fixed k,

$$q_n(k) \le \sum_{t=-\infty}^{\infty} |\beta_n(k,t)|^2 = C\int_{-\pi}^{\pi} |d_k(\lambda)\Psi(\lambda)\eta_n(\lambda)|^2 \mathrm{d}\lambda, \tag{5.27}$$

where $C > 0$ does not depend on k.

Let $d > 0$. By (6.19) of Lemma 6.4, we have that $q_n(k) = O(k_n^2|k|_+^{2\delta-1})$ if $k \le 0$, $q_n(k) = O(k_n^2|n-k+1|_+^{2\delta-1})$ if $1 \le k \le n$. Therefore

$$\sum_{k=-n+1}^{n} q_n(k) \le Ck_n^2\left(\sum_{k=-n+1}^{0} |k|_+^{2\delta-1} + \sum_{k=1}^{n} |n-k+1|^{2\delta-1}\right) = O(r_n^2(d,\delta)).$$

On the other hand, using bound (6.18) of Lemma 6.4, it follows that

$$\sum_{k=-\infty}^{-n} q_n(k) \le Ck_n^2 \sum_{k=-\infty}^{-n} n^{1-2d+2\delta}|k|^{2(d-1)} \le Ck_n^2 n^{2\delta} = O(r_n^2(d,\delta)),$$

to complete the proof of (5.19) in the case $d > 0$.

If $-1/2 < d < 0$, then proof of (5.19) for $i = 1$ follows the same lines as for $d > 0$ using estimates (6.18) and (6.19) of Lemma 6.4.

Finally we show (5.19) for $i = 2$. By (5.21),

$$E|V_{n,2} - EV_{n,2}|^2 \le C \sum_{k=-\infty}^{n} \sum_{t=1}^{n} |\zeta_n(k,t)|^2.$$

Then, as above in the case $i = 1$, by Parseval's equality and definition (5.13) of $\zeta_n(k,t)$ we obtain

$$z_n(k) := \sum_{t=1}^{n} |\zeta_n(k,t)|^2 \le \sum_{t=-\infty}^{\infty} |\zeta_n(k,t)|^2 \tag{5.28}$$

$$\le C \int_{-\pi}^{\pi} |c_n(\lambda)\Psi(\lambda)\eta_n(\lambda)|^2 \mathrm{d}\lambda \tag{5.29}$$

where $C > 0$ does not depend on k. In the case $d \ne 0$ by (6.20) of Lemma 6.4, $z_n(k) = O(k_n^2 n^{2\delta-1})$, uniformly in $1 \le k \le n$. Therefore

$$E|V_{n,2} - EV_{n,2}|^2 \le \sum_{k=1}^{n} z_n(k) \le Ck_n^2 n^{2\delta} = O(r_n^2(d,\delta)), \tag{5.30}$$

to prove (5.19).

(ii) Assume now that $d = 0$. By (5.22), we obtain that

$$E|Y_n|^2 \le C\left[\sum_{k,t=-\infty}^{0} |\nu_n(k,t)|^2 + \sum_{k,t=1}^{n} |\nu_n(k,t)|^2 + \sum_{k,t=1}^{n} |\mu_n(k,t)|^2\right] + \left[\left(\sum_{k=-\infty}^{0} |\nu_n(k,k)|\right)^2 + \left(\sum_{k=1}^{n} |\nu_n(k,k)|\right)^2 + \left(\sum_{k=1}^{n} |\mu_n(k,k)|\right)^2\right] = O(k_n^2)$$

since by Lemma 6.3, $|\nu_n(k,t)| \le Ck_n|k|_+^{-\alpha}|t|_+^{-\alpha}$ if $k,t \le 0$; $|\nu_n(k,t)| \le Ck_n|n-k+1|^{-\alpha}|n-t+1|_+^{-\alpha}$ if $1 \le k,t \le n$, and $|\mu_n(k,t)| \le Ck_n|n|^{-2\alpha}$ where $\alpha > 1$.

On the other hand, by (5.25) and (5.30),

$$E|V_{n,1} - EV_{n,1}|^2 \le C \sum_{k=-\infty}^{n} q_n(k) = O(k_n^2),$$

$$E|V_{n,2} - EV_{n,2}|^2 \le C \sum_{k=1}^{n} z_n(k) = O(k_n^2)$$

since by Lemma 6.4 below, $q_n(k) = O(k_n^2|k|_+^{-2\alpha})$ if $k \le 0$; $q_n(k) = O(k_n^2|n-k+1|_+^{-2\alpha})$ if $1 \le k \le n$ and $z_n(k) = O(k_n^2 n^{-2\alpha})$ if $1 \le k \le n$; with $\alpha > 1$.

These bounds together with (5.17) imply (5.1) in the case $d = 0$.

Proof of (5.2). We now assume only $EZ_j^2 < \infty$. Similarly to bound (5.17), we have

$$E\left|\int_{-\pi}^{\pi} L_n(\lambda)\eta_n(\lambda)\mathrm{d}\lambda\right| \le Cn^{-1}(E|Y_n| + E|V_{n,1}| + E|V_{n,2}|). \tag{5.31}$$

It thus suffices to show that

$$E|Y_n| = O(\bar{r}_n(d,\delta)); \qquad E|V_{n,i}| = O(\bar{r}_n(d,\delta)), \quad i = 1,2. \tag{5.32}$$

Estimates (5.20), (5.23) and (5.24) imply that $E|Y_n| \le Cm_n = O(r_n(d,\delta))$, and therefore Y_n satisfies (5.32), since $r_n(d,\delta) = O(\bar{r}_n(d,\delta))$.

Write

$$V_{n,1} = \sum_{k=-\infty}^{n} \sum_{t=1:t\ne k}^{n} \beta_n(k,t)Z_kZ_t + \sum_{k=1}^{n} \beta_n(k,k)Z_kZ_k =: V_{n,1}^{(1)} + V_{n,1}^{(2)}$$

and

$$V_{n,2} = \sum_{k=1}^{n} \sum_{t=1:t\neq k}^{n} \zeta_n(k,t) Z_k Z_t + \sum_{k=1}^{n} \zeta_n(k,k) Z_k Z_k =: V_{n,2}^{(1)} + V_{n,2}^{(2)}.$$

Then

$$E|V_{n,1}| \le E|V_{n,1}^{(1)}| + E|V_{n,1}^{(2)}|$$

where under finite second moment $EZ_j^2 < \infty$,

$$E|V_{n,1}^{(1)}| \le \left(E|V_{n,1}^{(1)}|^2\right)^{1/2} \le C \left(\sum_{k=-\infty}^{n} \sum_{t=1:t\neq k}^{n} |\beta_n(k,t)|^2 \right)^{1/2} = O(r_n(d,\delta))$$

as we have showed above estimating (5.25). On the other hand, it is easy to check that $E|V_{n,1}^{(2)}| \le C\sum_{k=1}^{n} |\beta_n(k,k)| = O(\bar{r}_n(d,\delta))$ using the bound (6.9) of $\beta_n(k,k)$. Thus $V_{n,1}^{(1)}$ satisfies (5.32). The proof for $V_{n,2}$ follows the same lines, but uses bound (6.10). ■

6. Auxiliary lemmas

Lemma 6.1. *If the coefficients ψ_j satisfy Assumption 2.1, then there is a constant C such that for any non-negative integers n_1, n_2, uniformly in $|\lambda| \le \pi$,*

$$\left| \sum_{j=n_1}^{n_2} \psi_j e^{i\lambda j} \right| \le C n_1^{d-1} |\lambda|^{-1}, \quad \text{if } |d| < 1/2 \text{ and } d \neq 0 \tag{6.1}$$

and

$$\left| \sum_{j=n_1}^{n_2} \psi_j e^{i\lambda j} \right| \le C \begin{cases} |\lambda|^{-d}, & \text{if } 0 < d < 1/2 \\ n_1^d, & \text{if } -1/2 < d < 0. \end{cases} \tag{6.2}$$

Moreover

$$|\psi(\lambda)| \le C|\lambda|^{-d}, \quad \text{for } |d| < 1/2. \tag{6.3}$$

Proof of Lemma 6.1. Note that $\left|\sum_{l=1}^{k} e^{i\lambda l}\right| = \frac{|\sin(\lambda k/2)|}{|\sin(\lambda/2)|} \le C|\lambda|^{-1}$ uniformly in $k \ge 1$. Let $-1/2 < d < 1/2$, $d \neq 0$. Then, using summation by parts,

$$\begin{aligned} \left| \sum_{j=n_1}^{n_2} \psi_j e^{i\lambda j} \right| &\le \sum_{j=n_1}^{n_2-1} |\psi_j - \psi_{j+1}| \left| \sum_{l=n_1}^{j} e^{i\lambda l} \right| + |\psi_{n_2}| \left| \sum_{l=n_1}^{n_2} e^{i\lambda l} \right| \\ &\le C \sum_{j=n_1}^{n_2-1} j^{d-2} |\lambda|^{-1} + C n_2^{d-1} |\lambda|^{-1} \le C n_1^{d-1} |\lambda|^{-1}, \end{aligned}$$

by Assumption 2.1, to prove (6.1).

To show (6.2) we consider first the case $0 < d < 1/2$. Set $n_0 = |\lambda|^{-1}$.

If $n_1 \le n_0 \le n_2$, then

$$\left|\sum_{j=n_1}^{n_2} \psi_j e^{i\lambda_j}\right| \le \left|\sum_{j=n_0}^{n_2} \psi_j e^{i\lambda_j}\right| + \left|\sum_{j=n_1}^{n_0} \psi_j e^{i\lambda_j}\right|$$
$$\le C n_0^{d-1}|\lambda|^{-1} + C\sum_{j=n_1}^{n_0} j^{-1+d} \le C n_0^{d-1}|\lambda|^{-1} + C n_0^d \le C|\lambda|^{-d}.$$

We have used above (6.1) to estimate the first sum and assumption $\psi_j \sim Cj^{-1+d}$ to estimate the second one.

If $n_0 = |\lambda|^{-1} < n_1$, then $n_1|\lambda| > 1$ and using (6.1), we obtain

$$\left|\sum_{j=n_1}^{n_2} \psi_j e^{i\lambda_j}\right| \le C n_1^{d-1}|\lambda|^{-1} \le C(n_1|\lambda|)^{-1+d}|\lambda|^{-d} \le C|\lambda|^{-d}.$$

If $n_0 = |\lambda|^{-1} > n_2$, then $|\sum_{j=n_1}^{n_2} \psi_j e^{i\lambda_j}| \le C\sum_{j=n_1}^{n_2} j^{-1+d} \le C n_2^d \le C|\lambda|^{-d}$.

Assume now that $-1/2 < d < 0$. Then $|\sum_{j=n_1}^{n_2} \psi_j e^{i\lambda_j}| \le C\sum_{j=n_1}^{n_2} j^{-1+d} \le C n_1^d$.

Finally, to show (6.3), note that in the case $d > 0$ (6.3) follows from (6.1), whereas for $d = 0$ we have that $|\psi(\lambda)| \le \sum_{t=0}^{\infty}|\psi_t| < \infty$ by (2.8). If $d < 0$ then $\psi_j = O(j^{-\alpha-1})$ with $\alpha = -d > 0$, by (2.7). It is well known that in this case function $\psi(\lambda) = \sum_{j=0}^{\infty}\psi_j e^{-i\lambda j}$ satisfies Lipschitz condition of order α, i.e. $|\psi(\lambda) - \psi(\lambda')| \le C|\lambda - \lambda'|^{\alpha}$, see [5], p. 210. Since by Assumption 2.1 we have that $\psi(0) = \sum_{j=0}^{\infty}\psi_j = 0$, this implies (6.3). ■

Lemma 6.2. *If the coefficients ψ_j satisfy Assumption 2.1, then there is a constant C such that for any non-negative integers n_1, n_2*

$$\left|\sum_{j=n_1}^{n_2} \psi_j e^{i\lambda j}\right| \le C\begin{cases} |\lambda|^{-d}(1+|n_1\lambda|)^{-(1-d)}, & \text{if } 0 < d < 1/2 \\ n_1^d(1+|n_1\lambda|)^{-1}, & \text{if } -1/2 < d < 0 \\ n_1^{-\alpha} & \text{if } d = 0. \end{cases} \tag{6.4}$$

Proof of Lemma 6.2. Consider first $d > 0$. In view of Lemma 6.1, it suffices to show that

$$|\lambda|^{-d}(1+|n_1\lambda|)^{-(1-d)} \ge (1/2)\min(n_1^{d-1}|\lambda|^{-1}, |\lambda|^{-d}).$$

This holds, because for $|\lambda n_1| \ge 1$, $|\lambda|^{-d}(1+|n_1\lambda|)^{-(1-d)} \ge |\lambda|^{-d}|n_1\lambda|^{-(1-d)}/2 = n_1^{d-1}|\lambda|^{-1}/2$ and for $|\lambda n_1| < 1$, $|\lambda|^{-d}(1+|n_1\lambda|)^{-(1-d)} \ge |\lambda|^{-d}/2$.

Suppose now that $d < 0$. Then, in view of Lemma 6.1, it suffices to show that

$$n_1^d(1+|n_1\lambda|)^{-1} \ge (1/2)\min(n_1^{d-1}|\lambda|^{-1}, n_1^d).$$

This holds, because for $|\lambda n_1| > 1$, $n_1^d(1+|n_1\lambda|)^{-1} \ge n_1^d(2|n_1\lambda|)^{-1} = n_1^{d-1}|\lambda|^{-1}/2$ and for $|\lambda n_1| \le 1$, $n_1^d(1+|n_1\lambda|)^{-1} \ge n_1^d/2$.

In the case $d = 0$, $\left|\sum_{j=n_1}^{n_2} \psi_j e^{i\lambda j}\right| \le \sum_{j=n_1}^{\infty}|\psi_j| \le C n_1^{-\alpha}$ by Assumption 2.1 to prove (6.4). ■

Corollary 6.1. *Under Assumption 2.1,*

$$|d_k(\lambda)| \le C \begin{cases} |\lambda|^{-d}(1+|k|_+|\lambda|)^{-(1-d)}, & \text{if } k \le 0 \text{ and } d > 0,\\ |k|_+^{d}(1+|k|_+|\lambda|)^{-1}, & \text{if } k \le 0 \text{ and } d < 0,\\ |\lambda|^{-d}(1+(n-k+1)|\lambda|)^{-(1-d)}, & \text{if } 1 \le k \le n \text{ and } d > 0,\\ (n-k+1)^{d}(1+(n-k+1)|\lambda|)^{-1}, & \text{if } 1 \le k \le n \text{ and } d < 0,\\ |k|_+^{-\alpha}, & \text{if } k \le 0 \text{ and } d = 0,\\ (n-k+1)^{-\alpha}, & \text{if } 1 \le k \le n \text{ and } d = 0 \end{cases} \tag{6.5}$$

and

$$|c_n(\lambda)| \le C \begin{cases} |\lambda|^{-d}(1+n|\lambda|)^{-(1-d)}, & \text{if } d > 0,\\ n^{d}(1+n|\lambda|)^{-1}, & \text{if } d < 0,\\ n^{-\alpha}, & \text{if } d = 0. \end{cases} \tag{6.6}$$

Lemma 6.3. *Suppose that assumptions of Theorem 2.1 are satisfied and (5.5) and (5.6) hold. Then the following bounds are valid uniformly in t, k:*

$$|\nu_n(k,t)| \le Ck_n \begin{cases} (6.7.1)\ |k|_+^{(\delta-1)/2}|t|_+^{(\delta-1)/2}, & \text{if } k,t \le 0 \text{ and } d \ne 0,\\ (6.7.2)\ (n-k+1)^{(\delta-1)/2}(n-t+1)^{(\delta-1)/2}, & \text{if } 1 \le k,t \le n \text{ and } d \ne 0,\\ (6.7.3)\ n^{1+\beta}|k|^{d-1}|t|^{d-1}, & \text{if } k,t \le -n \text{ and } d > 0,\\ (6.7.4)\ n^{1+\beta}|k|^{d-1}|t|^{d-1} + n^{\beta}|k|^{d-1/2}|t|^{d-1/2}, & \text{if } k,t \le -n \text{ and } d < 0,\\ (6.7.5)\ |k|_+^{-\alpha}|t|_+^{-\alpha}, & \text{if } k,t \le 0 \text{ and } d = 0,\\ (6.7.6)\ (n-k+1)^{-\alpha}(n-t+1)^{-\alpha}, & \text{if } 1 \le k,t \le n \text{ and } d = 0, \end{cases} \tag{6.7}$$

$$|\mu_n(k,t)| \le Ck_n \begin{cases} n^{2\delta-1}, & \text{for } k \le n \text{ and } d \ne 0,\\ n^{-2\alpha}, & \text{for } k \le n \text{ and } d = 0, \end{cases} \tag{6.8}$$

$$|\beta_n(k,t)| \le Ck_n \begin{cases} (n-k+1)^{\delta-1}, & \text{if } 1 \le k \le n \text{ and } d < 0,\\ |k|_+^{\delta-1}, & \text{if } k \le 0 \text{ and } d < 0,\\ (n-k+1)^{\max(\delta,d)-1}, & \text{if } 1 \le k \le n, \delta \ne d \text{ and } d > 0,\\ (n-k+1)^{\delta-1}\log k, & \text{if } 1 \le k \le n, \delta = d \text{ and } d > 0,\\ (n-k+1)^{-\alpha}, & \text{if } 1 \le k \le n \text{ and } \delta = 0. \end{cases} \tag{6.9}$$

and

$$|\zeta_n(k,t)| \le Ck_n \begin{cases} n^{\delta-1}, & \text{if } 1 \le k \le n \text{ and } d < 0,\\ n^{\max(\delta,d)-1}, & \text{if } \delta \ne d \text{ and } d > 0,\\ n^{\delta-1}\log n, & \text{if } \delta = d \text{ and } d > 0,\\ n^{-\alpha}, & \text{if } \delta = 0. \end{cases} \tag{6.10}$$

Proof of Lemma 6.3. *Proof of (6.7).* We first establish (6.7.1) assuming $d \in (0, 1/2)$. Using Corollary 6.1, we obtain

$$|d_k(\lambda)| \le C|\lambda|^{-d}(1+|k|_+|\lambda|)^{-(1-d)},$$

which together with Assumption 2.2 and definition $\delta = \beta + 2d$ implies that

$$|\nu_n(k,t)| \le Ck_n \int_{-\pi}^{\pi} \frac{|\lambda|^{-d}}{(1+|k|_+|\lambda|)^{1-d}} \frac{|\lambda|^{-d}}{(1+|t|_+|\lambda|)^{1-d}} |\lambda|^{-\beta} \mathrm{d}\lambda$$

$$\le Ck_n\left(\int_{-\pi}^{\pi}\frac{|\lambda|^{-\delta}}{(1+|k|_+|\lambda|)^{2-2d}}\mathrm{d}\lambda\right)^{1/2}\left(\int_{-\pi}^{\pi}\frac{|\lambda|^{-\delta}}{(1+|t|_+|\lambda|)^{2-2d}}\mathrm{d}\lambda\right)^{1/2}$$
$$\le Ck_n|k|_+^{(\delta-1)/2}|t|_+^{(\delta-1)/2}.$$

Noting that $\delta+2-2d=2+\beta>1$, because $|\beta|<1$ and observing that $2-2d>1$ because $|d|<1/2$, the last bound follows using the estimate

$$\int_{-\pi}^{\pi}\frac{|\lambda|^{-\alpha_1}}{(1+|n\lambda|)^{\alpha_2}}\mathrm{d}\lambda\le n^{\alpha_1+\alpha_2-1}\int_{-\infty}^{\infty}\frac{|u|^{-\alpha_1}}{(1+|u|)^{\alpha_2}}\mathrm{d}u\le Cn^{\alpha_1+\alpha_2-1},\tag{6.11}$$

where $\alpha_1>0,\alpha_2<1,\alpha_1+\alpha_2>1$, which follows on setting $u=n\lambda$.

Next we show (6.7.1) in the case $d\in(-1/2,0)$. By the same arguments,

$$|v_n(k,t)|\le Ck_n\int_{-\pi}^{\pi}\frac{|k|_+^d}{(1+|k|_+|\lambda|)}\frac{|t|_+^d}{(1+|k|_+|\lambda|)}|\lambda|^{-\beta}\mathrm{d}\lambda$$
$$\le Ck_n\left(\int_{-\pi}^{\pi}\frac{|k|_+^{2d}|\lambda|^{-\beta}}{(1+|k|_+|\lambda|)^2}\mathrm{d}\lambda\right)^{1/2}\left(\int_{-\pi}^{\pi}\frac{|t|_+^{2d}|\lambda|^{-\beta}}{(1+|k|_+|\lambda|)^2}\mathrm{d}\lambda\right)^{1/2}$$
$$\le C|k|_+^{(\delta-1)/2}|t|_+^{(\delta-1)/2}.$$

Proof of (6.7.2) follows using the same argument as above.

To show (6.7.3) in the case $0<d<1/2$, write $|v_n(k,t)|\le I_1+I_2$, where

$$I_1=\int_{|\lambda|<\pi/n}|d_k(\lambda)d_t(\lambda)||\eta_n(\lambda)|\mathrm{d}\lambda,\qquad I_2=\int_{\pi/n\le|\lambda|\le\pi}|d_k(\lambda)d_t(\lambda)||\eta_n(\lambda)|\mathrm{d}\lambda.$$

Since for $k\le -n$,

$$|d_k(\lambda)|\le\sum_{j=1-k}^{n-k}|\psi_j|\le C\sum_{j=1-k}^{n-k}|j|^{-1+d}\le Cn|k|^{d-1},\tag{6.12}$$

Assumption 2.2 implies that

$$I_1\le Ck_n\int_0^{\pi/n}n|k|^{d-1}n|t|^{d-1}\lambda^{-\beta}\mathrm{d}\lambda$$
$$\le Ck_nn^2|k|^{d-1}|t|^{d-1}(\pi/n)^{-\beta+1}\le Ck_n|k|^{d-1}|t|^{d-1}n^{1+\beta}.\tag{6.13}$$

Similarly, by (6.1),

$$I_2\le C\int_{\pi/n}^{\pi}|k|^{d-1}\lambda^{-1}|t|^{d-1}\lambda^{-1}\lambda^{-\beta}\mathrm{d}\lambda\le C|k|^{d-1}|t|^{d-1}\int_{\pi/n}^{\pi}\lambda^{-2-\beta}\mathrm{d}\lambda$$
$$\le C|k|^{d-1}|t|^{d-1}n^{1+\beta}.\tag{6.14}$$

To show (6.7.4) for $-1/2<d<0$ and $t,k\le -n$, estimation of I_2 is the same as in the verification of (6.13). It gives the bound $|I_2|\le n^{1+\beta}|k|^{d-1}|t|^{d-1}$. Estimation of I_1 is different and it gives the bound $|I_1|\le n^{\beta}|k|^{d-1/2}|t|^{d-1/2}$. To derive it, we use the bound

$$|d_k(\alpha)|\le\sum_{j=-k+1}^{n-k}|\psi_j|\le n^{1/2}\left(\sum_{j=-k+1}^{n-k}\psi_j^2\right)^{1/2}$$

$$\le C n^{1/2} \left(\sum_{j=-k+1}^{n-k} |j|^{2d-2} \right)^{1/2} \le C n^{1/2} |k|^{d-1/2} \tag{6.15}$$

and a similar argument as in the verification of (6.7.2).

To show (6.7.5) and (6.7.6) for $d = 0$ note that assumption (2.8) on ψ_j implies that

$$|d_k(\lambda)| \le C|k|_+^{-\alpha}, \quad k \le 0; \qquad |d_k(\lambda)| \le C|n-k+1|_+^{-\alpha}, \quad 1 \le k \le n.$$

These bounds together with definition (5.10) of $\nu_n(k,t)$ imply (6.7.5) and (6.7.6).

Proof of (6.8). Since $|\mu_n(k,t)| \le C k_n \int_{-\pi}^{\pi} |c_n(\lambda)|^2 |\lambda|^{-\beta} \mathrm{d}\lambda$ bound (6.8) can be established similarly as bound (6.7.1) applying to $c_n(\lambda)$ bound (6.6).

Proof of (6.9). Note that by (2.30),

$$|\Psi(\lambda)| \le C f^{1/2}(\lambda) \le C|\lambda|^{-d}, \quad \lambda \in [-\pi, \pi]. \tag{6.16}$$

If $d < 0$, then for $1 \le k \le n$ using (6.16), Assumption 2.2 and (6.4), we obtain

$$\begin{aligned} |\beta_n(k,t)| &\le \int_{-\pi}^{\pi} |d_k(\lambda)\Psi(\lambda)\eta_n(\lambda)| \mathrm{d}\lambda \le C k_n \int_{-\pi}^{\pi} |d_k(\lambda)||\lambda|^{-d-\beta} \mathrm{d}\lambda \\ &\le C k_n |n-k+1|^d \int_{-\pi}^{\pi} \frac{|\lambda|^{-d-\beta}}{1+|n-k+1||\lambda|} \mathrm{d}\lambda \\ &\le C k_n |n-k+1|^{2d+\beta-1} = C k_n |n-k+1|^{\delta-1}, \end{aligned} \tag{6.17}$$

using (6.11) and noting that $d + \beta = \delta - d < 1$ and $1 + d + \beta > 1$ by (5.6).

If $d < 0$ and $k \le 0$, then (6.9) follows by the same argument as above using (6.5).

If $d > 0$, then by (6.4),

$$|\beta_n(k,t)| \le C k_n \int_{-\pi}^{\pi} \frac{|\lambda|^{-2d-\beta}}{(1+|n-k+1||\lambda|)^{1-d}} \mathrm{d}\lambda.$$

Then, if $\delta > d$ applying (6.11), we obtain

$$|\beta_n(k,t)| \le C k_n |n-k+1|^{\delta-1} \int_{-\infty}^{\infty} \frac{|\lambda|^{-\delta}}{(1+|\lambda|)^{1-d}} \mathrm{d}\lambda \le C k_n |n-k+1|^{\delta-1},$$

whereas if $\delta < d$, then

$$|\beta_n(k,t)| \le C k_n |n-k+1|^{-1+d} \int_{-\pi}^{\pi} |\lambda|^{-1-(\delta-d)} \mathrm{d}\lambda \le C k_n |n-k+1|^{d-1},$$

and finally, for $\delta = d$,

$$|\beta_n(k,t)| \le C k_n |n-k+1|^{\delta-1} \int_{-\pi k}^{\pi k} \frac{|\lambda|^d}{(1+|\lambda|)^{1-d}} \mathrm{d}\lambda \le C k_n |n-k+1|^{\delta-1} \log k,$$

to prove (6.9).

If $d = 0$ then $|d_k(\lambda)| \le C|n-k+1|_+^{-\alpha}$, $1 \le k \le n$ and

$$|\beta_n(k,t)| \le C k_n |n-k+1|_+^{-\alpha} \int_{-\pi}^{\pi} |\lambda|^{-d-\beta} d \le C k_n |n-k+1|_+^{-\alpha}$$

because $d + \beta = \delta - d < 1$, to prove (6.9).

Proof of (6.10). Using bound (6.6) on $c_n(\lambda)$, the proof of (6.10) is the same as the proof of (6.9). ■

Lemma 6.4. *Suppose the assumptions of Lemma 6.3 hold. The following bounds hold uniformly in $k \le n$ and $n \ge 1$ for the quantities $q_n(k)$ (5.26) and $z_n(k)$ (5.28):*

$$q_n(k) \le Ck_n^2 \begin{cases} |k|^{-2+2d} n^{1-2d+2\delta}, & \text{if } k \le -n \text{ and } d > 0, \\ |k|^{-2+2\delta} n, & \text{if } k \le -n \text{ and } d < 0, \\ |k|_+^{-2\alpha} & \text{if } k \le 0 \text{ and } d = 0, \\ |n-k+1|^{-2\alpha} & \text{if } 1 \le k \le n \text{ and } d = 0, \end{cases} \tag{6.18}$$

$$q_n(k) \le Ck_n^2 \begin{cases} |k|_+^{2\delta-1}, & \text{for } k \le 0 \text{ and } d > 0, \\ |k|_+^{2\delta-1}, & \text{for } k \le 0 \text{ and } d < 0, \delta - d < 1/2, \\ \delta_n(k), & \text{for } k \le 0 \text{ and } d < 0, \delta - d \ge 1/2, \\ |n-k+1|^{2\delta-1}, & \text{for } 1 \le k \le n \text{ and } d > 0, \\ |n-k+1|^{2\delta-1}, & \text{for } 1 \le k \le n \text{ and } d < 0, \delta - d < 1/2, \\ \delta_n(n+1-k), & \text{for } 1 \le k \le n \text{ and } d < 0, \delta - d \ge 1/2 \end{cases} \tag{6.19}$$

where

$$\delta_n(k) = |k|_+^{2d} n^{-1+2\delta-2d} + |k|_+^{\epsilon-1} n^{2\delta-\epsilon}$$

with some $\epsilon \in (0, 1)$, and

$$z_n(k) \le Ck_n^2 \begin{cases} n^{2\delta-1} & \text{if } 1 \le k \le n \text{ and } d \ne 0, \\ n^{-2\alpha} & \text{if } 1 \le k \le n \text{ and } d = 0. \end{cases} \tag{6.20}$$

Proof of Lemma 6.4. *Proof of* (6.18). Let $d > 0$ and $k < 0$. From definition (5.27) of $q_n(k)$, together with Assumption 2.2 and (6.16), it follows that

$$q_n(k) \le Ck_n^2 \int_{-\pi}^{\pi} |d_k(\lambda)|^2 |\lambda|^{-2d-2\beta} \mathrm{d}\lambda = Ck_n^2 (J_1 + J_2) \tag{6.21}$$

where

$$J_1 = \int_{|\lambda| \le 1/n} |d_k(\lambda)|^2 |\lambda|^{-2d-2\beta} \mathrm{d}\lambda, \qquad J_2 = \int_{1/n < |\lambda| \le \pi} |d_k(\lambda)|^2 |\lambda|^{-2d-2\beta} \mathrm{d}\lambda.$$

In case $d > 0$, using the estimate (6.12) and observing that $2d + 2\beta = 2\delta - 2d < 1$, we obtain

$$J_1 \le n^2 |k|^{-2+2d} \int_{|\lambda| \le 1/n} |\lambda|^{2d-2\delta} \mathrm{d}\lambda \le C|k|^{-2+2d} n^{1-2d+2\delta}.$$

Next, using bound (6.1) of Lemma 6.1 we see that

$$J_2 \le C|k|^{-2(1-d)} \int_{1/n \le |\lambda| \le \pi} |\lambda|^{-2-2d-2\beta} \mathrm{d}\lambda \le C|k|^{-2+2d} n^{1-2d+2\delta}$$

noticing that $1 + 2d + 2\beta = 1 + 2\delta - 2d > 0$, by assumption (5.5).

To prove (6.18) for $d < 0$ and $k < 0$, we bound $|\beta_n(k,t)| \le Ck_n |k|^{\delta-1}$ by estimate (6.9), which yields

$$q_n(k) \le C \sum_{t=1}^{n} |\beta_n(k,t)|^2 \le Ck_n^2 |k|_+^{2\delta-2} n.$$

If $d = 0$, then by Corollary 6.1, $|d_k(\lambda)| \leq C|k|_+^{-\alpha}$ for $k \leq 0$, which implies that

$$q_n(k) \leq Ck_n^2 |k|^{-2\alpha} \int_{-\pi}^{\pi} |\lambda|^{-2\beta} \mathrm{d}\lambda \leq Ck_n^2 |k|^{-2\alpha}$$

since $d = 0$ and therefore $2\beta = 2\delta < 1$. In the case $1 \leq k \leq n$ the proof of (6.18) is similar. This completes the proof of (6.18).

Proof of (6.19). We shall prove (6.19) only in the case $k \leq 0$ since for $1 \leq k \leq n$ the proof follows using the same argument.

Assume that $d > 0$. Then estimating $d_k(\lambda)$ by (6.4) of Lemma 6.2 we obtain that

$$\begin{aligned} q_n(k) &\leq Ck_n^2 \int_{-\pi}^{\pi} \frac{|\lambda|^{-4d-2\beta}}{(1+|k||\lambda|)^{2(1-d)}} \mathrm{d}\lambda \leq Ck_n^2 \int_{-\pi}^{\pi} \frac{|\lambda|^{-2\delta}}{(1+|k||\lambda|)^{2(1-d)}} \mathrm{d}\lambda \\ &\leq Ck_n^2 |k|^{2\delta-1} \int_{-\infty}^{\infty} \frac{|\lambda|^{-2\delta}}{(1+|\lambda|)^{2(1-d)}} \mathrm{d}\lambda \leq C|k|^{2\delta-1} \end{aligned} \tag{6.22}$$

by (6.11), since $2\delta < 1$ and $2 - 2d + 2\delta > 1$, because $1 - 2d + 2\delta = 1 + 2d + 2\beta > 0$ by assumption (5.5).

Assume that $d < 0$ and $\delta - d < 1/2$. Then estimating $d_k(\alpha)$ by (6.4) we obtain

$$\begin{aligned} q_n(k) &\leq Ck_n^2 \int_{-\pi}^{\pi} \frac{|k|_+^{2d} |\lambda|^{-2d-2\beta}}{(1+|k||\lambda|)^2} \mathrm{d}\lambda \\ &\leq Ck_n^2 ||k_+|^{4d+2\beta-1} \int_{-\infty}^{\infty} \frac{|\lambda|^{-2d-2\beta}}{(1+|\lambda|)^2} \mathrm{d}\lambda \leq Ck_n^2 |k|^{2\delta-1} \end{aligned}$$

by (6.11), since $2d + 2\beta = 2(\delta - d) < 1$ and $2 + 2d + 2\beta > 1$, by assumption (5.5).

Assume that $d < 0$ and $2\delta - 2d \geq 1$. Since $|2d| < 1$ this implies that $2\delta > 0$ and therefore there exists $\epsilon > 0$ such that

$$2\delta - \epsilon > 0, \qquad 2d - \epsilon > -1. \tag{6.23}$$

Then we can write $\beta_n(k, t)$ as

$$\beta_n(k,t) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(t-k)\lambda} d_k(\lambda) \Psi(\lambda) \eta_n(\lambda) \mathrm{d}\lambda = \beta_n^-(k,t) + \beta_n^+(k,t) \tag{6.24}$$

where

$$\beta_n^-(k,t) = \int_{|\lambda| \leq 1/n} [\cdots] \mathrm{d}\lambda, \qquad \beta_n^+(k,t) = \int_{1/n < |\lambda| \leq \pi} [\cdots] \mathrm{d}\lambda.$$

Hence

$$q_n(k) \leq 2 \left(\sum_{t=1}^{n} |\beta_n^-(k,t)|^2 + \sum_{t=1}^{n} |\beta_n^+(k,t)|^2 \right).$$

By (6.2),

$$\begin{aligned} |\beta_n^-(k,t)| &\leq \int_{|\lambda| \leq 1/n} |d_k(\lambda) \Psi(\lambda) \eta_n(\lambda)| \mathrm{d}\lambda \leq Ck_n |k|_+^d \int_{|\lambda| \leq 1/n} |\lambda|^{-d-\beta} \mathrm{d}\lambda \\ &\leq Ck_n |k|_+^d n^{-1+d+\beta} \end{aligned}$$

since $d+\beta=\delta-d<1$. Therefore,

$$\sum_{t=1}^{n}|\beta_n^-(k,t)|^2 \le Ck_n^2 n^{-1+2d+2\beta}|k|_+^{2d} = Ck_n^2 n^{-1+2\delta-2d}|k|_+^{2d}.$$

On the other hand, applying Parseval's equality and then (6.4), similarly as in (5.27) we obtain

$$\begin{aligned}\sum_{t=1}^{n}|\beta_n^+(k,t)|^2 &\le C\int_{1/n\le|\lambda|\le\pi}|d_k(\lambda)\Psi(\lambda)\eta_n(\lambda)|^2\mathrm{d}\lambda\\ &\le Ck_n^2|k|_+^{2d}\int_{1/n\le|\lambda|\le\pi}\frac{1}{(1+|k_+||\lambda|)^2}\lambda^{-2d-2\beta}\mathrm{d}\lambda.\end{aligned}$$

Let $\epsilon>0$ satisfies (6.23). Since $2d+2\beta=2\delta-2d$, we can estimate $|\lambda|^{-2\delta+\epsilon}\le n^{2\delta-\epsilon}$ when $|\lambda|\ge 1/n$. Thus

$$\sum_{t=1}^{n}|\beta_n^+(k,t)|^2 \le Ck_n^2|k|_+^{2d}n^{2\delta-\epsilon}\int_{0\le|\lambda|\le\pi}\frac{\lambda^{2d-\epsilon}}{(1+|k_+||\lambda|)^2}\mathrm{d}\lambda \le Ck_n^2|k|_+^{\epsilon-1}n^{2\delta-\epsilon}$$

by (6.11), since $-2d+\epsilon<1$ and $2-2d+\epsilon>1$. Thus

$$\begin{aligned}q_n(k) &\le C\left(\sum_{t=1}^{n}|\beta_n^-(k,t)|^2+\sum_{t=1}^{n}|\beta_n^+(k,t)|^2\right)\\ &\le Ck_n^2(|k|_+^{2d}n^{-1+2\delta-2d}+|k|_+^{\epsilon-1}n^{2\delta-\epsilon})\equiv C\delta_n(k),\end{aligned}$$

to prove (6.19).

Proof of (6.20). Assume that $d>0$. Then from definition (5.29) using Assumption 2.1 and (6.16) it follows that

$$z_n(k)\le Ck_n^2\int_{-\pi}^{\pi}|c_n(\lambda)|^2|\lambda|^{-2d-2\beta}\mathrm{d}\lambda. \tag{6.25}$$

Since by (6.4), $|c_n(\lambda)|\le|\lambda|^{-d}(1+n|\lambda|)^{-1+d}$ then the estimate (6.20) follows using the same argument as in (6.22).

Let $d<0$. Then by (6.4) $|c_n(\lambda)|\le n^{-d}(1+n|\lambda|)^{-1}$. Therefore by the argument used in estimating $\beta_n(k,t)$ in (6.24), it follows that $z_n(k)\le\sum_{t=1}^{n}|\zeta_n(k,t)|^2\le Ck_n^2n^{2\delta-1}$, to prove (6.20).

If $d=0$, then by (6.4), $|c_n(\lambda)|\le n^{-\alpha}$. Then the bound $z_n(k)\le Ck_n^2n^{-2\alpha}$ follows from (6.25), noting that in the case $d=0$, $2\delta=2\beta<1$, to complete the proof of (6.20). ■

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